

# Nuts, Fish, Babies, and Nuts: Modeling with Cubic and Nearly-Cubic Functions

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**Abstract:** In this chapter, we reflect on an example from Engel (2010) in which he models how the weight of hexnuts depends on their size. Based on our understanding of scaling, we expect the relationship to be cubic, and it is. We then explore examples that are nearly cubic, using data on snook (a kind of fish), where the power-law exponent is greater than 3, and human children, where the exponent is less than 3. On our return to the hexnuts, we find that the exponent is not 3 after all. Further analysis shows that in fact the relationship is not even a power law. Along the way we make observations about the role of data in generating insight, and the meaning of fit parameters such as the non-integer exponents. Finally, we reflect on how these problems illuminate our understanding of the nature of modeling as a whole.

## 1 Modeling with cubics

In *Anwendungsorientierte Mathematik: von Daten zur Funktion*, Engel (2010) investigates how the weight of hexnuts depends on their diameter. In this chapter, we will meet these data again and see how the conclusions we might draw from them apply—and do not apply—to other data sets.

Of course, we are not professionally interested in the weights of hexnuts. We're interested in the pedagogy of modeling. Using a cubic to describe the relationship of diameter to weight is an example of mathematical modeling, and we want our students to engage in a wide variety of modeling activities—especially when, as is the case here, the functional form of the model makes logical, geometrical sense, and helps illuminate the context.

### 1.1 Reviewing the Hexnuts Data

Let us begin by reviewing the original hexnut data and its analysis. The author measured and weighed a number of hexnuts. The *size* of the hexnut is the diameter of the bolt it fits. The face-to-face “width” of the nut we will call *face*, which is larger than *size* (see Figure 1, left).

If the nuts are geometrically similar—and made of the same material—our understanding of dimensionality and scaling suggests that the relationship between size and weight should be *cubic*.

## 1.2 Pedagogical Path

But not all students will have that understanding at first. They will see that the data do not lie on a straight line. But when asked what function could fit the points, many will first suggest using a parabola.

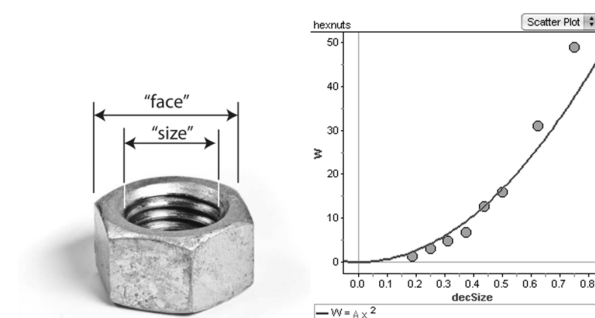


Figure 1. On the left, a hexnut showing nomenclature for this paper. The *size* of the nut is the diameter of the bolt it fits. The measurement from one flat face to the opposite side we will call *face*. On the right, a poor quadratic model for the original hexnut data (*decSize* is decimal size, in inches). The parameter  $A$  is about 65.

Using  $W$  for the weight and  $x$  for the linear size of the nut  $W(x) = x^2$ , is a terrible fit. The model needs a coefficient, so students try  $W(x) = Ax^2$ . Using Fathom<sup>1</sup> they can make a slider for  $A$  and vary its value to improve the fit. This works better, but still not very well (Figure 1, right).

It seems that no value for the parameter  $A$  will make the curve fit the data. A quadratic does not make a good model; we should look for a different function.

Let's stop here and examine how much the student must understand in order to get to this point. We have rushed through the reasoning as if it's easy, but in order

<sup>1</sup> A number of other tools these days will serve as well, including the Desmos graphing calculator, <http://www.desmos.com/calculator>.

to be a competent modeler, a student has to develop considerable skills and understandings, only some of which are part of the traditional math curriculum.

For example:

- How do you learn that including a coefficient gives you power over the shape of the function?
- How do you know that a coefficient will change the function in a useful way? That is, why is *that* parameter so useful? Why not add a constant?
- How good a fit is good enough? Specifically, how do we know that the function in Figure 1 is *not* good enough?
- How can you tell that no value for a parameter will make the curve fit? (This is more difficult with multiple parameters, of course.)
- How do you come to understand that if you can't make the curve fit, there is something wrong with the *form* of the function you're using?
- If the form of the function is wrong, how do you find a better one?

This list is a good start at cataloging some of the questions we must answer if we want to include more modeling in the math curriculum. We can even make a start at answering some of them.

### 1.3 Assessing fit: use residual plots

For example, how do we know that quadratic fit is bad? We can see in Figure 1 that the data are below the function at first, but above the function later. Teaching students about *residual plots* is a good way to address this issue of fit quality. If students remember to look at a residual plot, they can apply simple rules. If a fit is good, points in the residual plot:

- are centered around zero;
- show no particular trend; and
- (less importantly) show about the same variation over the domain.

In the case of the quadratic, these residual plots—made with different values of the parameter  $A$ —show clear trends and departures from zero:

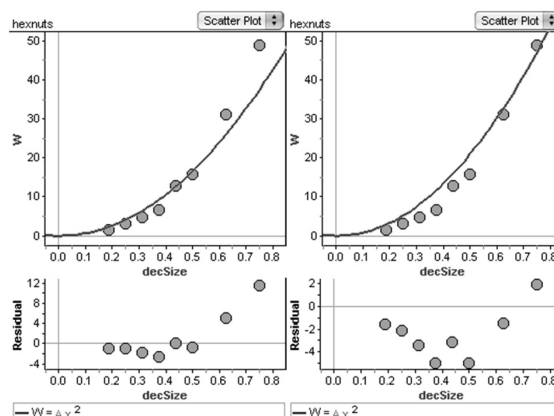


Figure 2. Two different quadratics, showing the residual plots below the main plots.

### 1.4 Finding a better functional form

If no quadratic will work, how do we find a better function? If we had had the insight about how the nuts are similar, and knew about scaling, we might have picked a cubic from the beginning. But if we didn't have that insight, we could still alter the function and see what happens. There are at least two strategies:

- If we know about the shapes of functions, we can pick a different one that has the same overall shape. In this example, if we knew that an exponential can be concave up (like the data) we might try such a function.
- We can take the function we have, and create a new parameter, that is, vary something that was once constant. In this case, if we vary the *exponent*, we will find a good model.

This last strategy creates the problem of dealing with two parameters. Typically students alternate from one to the other, making successive approximations. In some situations this is terribly inefficient if done blindly. But by taking a broader view, students can develop better “fit-by-eye” strategies. For example,

- Change the exponent slider until the function looks like the right *shape*, then change the “stretch” coefficient  $A$ .
- Watch the residuals. In some situations, changing parameters has a systematic effect on the residuals that's easier to see than in the main plot.

Either of these strategies lets students witness the effects of changing parameters and helps them learn to reason about the functions.

### 1.5 Models lead to insight

Once students find a function, a good fit—for example,  $W(x) = 111x^{2.93}$ —they can learn to ask, *is it possible that the exponent is actually 3.0?* Then, discovering that the cubic fit is about as good as the one with the variable exponent, two things happen:

- They can get rid of a parameter, applying Occam’s Razor.
- They can wonder, is there any underlying reason why the parameter should be exactly 3?

This “answer analysis<sup>2</sup>” can lead students to insights about scaling. That is, (and this claim I put forward with only anecdotal evidence) *some students can discover geometrical relationships through measurement and data.*

The naïve educator expects the opposite: that students should reason about length and volume, and predict a cubic relationship. But some students do not succeed on this path. Since we have the technology to make it easy to model with functions, why not let them find the cubic relationship, and then use that “answer” to guide them to the underlying reason?

## 2 Beyond Hexnuts: Exponents greater than 3

Hexnuts are quite uniform in their manufacture. Let us apply our insight about volume and cubic functions to other data.

### 2.1 Alligators

Later in Engel’s book (page 269), he presents a famous data set containing the lengths and weights of a number of alligators caught in Florida. The object is to assess the weight of alligators without actually catching them; one can estimate length from aerial photographs, but weighing an alligator can be hard.

The relationship is roughly cubic, but there are not very many gators in the data set. The skeptical reader may worry, with good reason, whether the two large

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<sup>2</sup> This is a familiar phenomenon to any problem-solver: If you are told the answer, it can often lead you to the insight you would have needed to find the answer on your own.

alligators are representative of large alligators in general. It would be great to have more data.

## 2.2 Snook

The author, therefore, telephoned the Florida Fish and Wildlife Conservation Commission, asking if there were more alligator data. The (amused and perplexed) representative was unaware of any, but provided data on thousands of snook (a type of fish) caught off the Florida coasts.

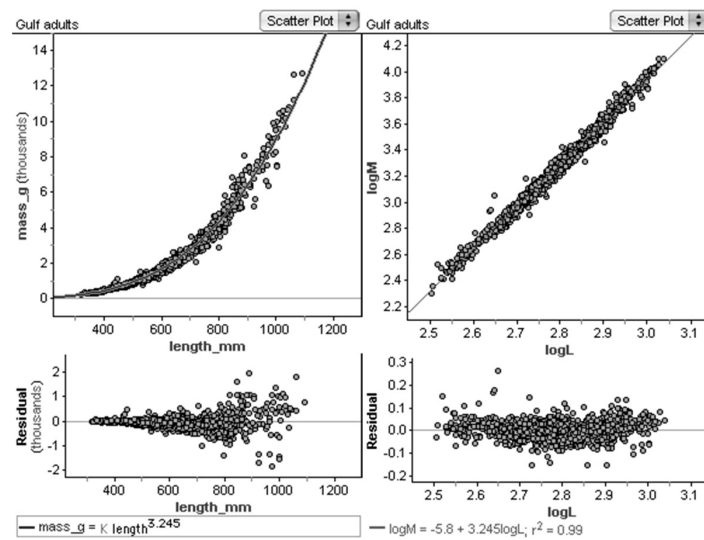


Figure 3. 1202 Gulf Snook over 30 cm in length. On the left, the mass-length plot; on the right, the same data, plotting the logarithms of those quantities. The slope of the least-squares line in that plot, 3.245, is the exponent in the back-transformed model—the curve in the left-hand graph.

In the data set, these snook range in length from 17 cm to well over 1 meter, and have weights ranging from about 50 grams to over 17 kg. Figure 3 shows a subset of the data: Snook caught on the Gulf coast (i.e., west of Florida in the Gulf of Mexico) and over 30 cm in length.

Engel introduces a log-log transformation in the alligator analysis and shows how useful it is for modeling a power law. We can use a slider for a variable exponent or fit the log-log data with a least-squares line. Its slope is the exponent. One advantage of transforming the data is that the least-squares process eliminates the need for messing with sliders. A disadvantage is that it is harder to picture the fish based on the data.

It seems that a power law with an *exponent greater than three* makes a good model for these data. How do we make sense of that?

First, let's remember why we thought the exponent would be three in the first place. (This is a good question for students. By the time they have a non-integer exponent for their model function, they will have forgotten.) That was because if these fish—or hexnuts, or alligators—are different sizes *but the same shape*, a fish twice as long will have eight times the volume and, presumably, eight times the weight. In general, the weight will be proportional to the *cube* of the length precisely because we assume the fish's shape has been stretched by the same proportion in *three* dimensions.

Put another way, if the fish are geometrically similar, the weight-length relationship will be cubic.

Since it is not, we have a strong result: the fish are *not geometrically similar*.

In what way are they not similar? Our data tell us: the big fish, like the big alligators, are heavier than we would expect if the fish were all the same shape. Therefore, the big fish are chubbier, fatter, plumper—more spherical, if you will—than their smaller, sleeker, skinnier counterparts.

Note that we can use that exponent as a measure of the way the shape changes as the objects get larger. If the fish grew like a train, getting longer without getting thicker, weight would be proportional to length, and the exponent would be 1.0. The long ones would be skinnier, in proportion, than the short ones. If they grew longer and “taller” without getting thicker—more like a pancake—the exponent would be 2.0.

### 3 Exponents less than 3

Thus these snook, with an exponent of 3.245, have “chubbier” shapes the larger they get. Can we think of a creature that starts out chubby and becomes less so as they get larger?

Of course we can.

The USA Centers for Disease Control does an extensive health survey—of humans—and publishes the data. We extracted height and weight data for 1147 children aged 0 to 48 months.

We look at the data before we take the logarithms to see whether cubics make any sense, or quadratics, or whatever is in between. Looking at residual plots, for example, it's easy to see that a cubic is too steep.

Then we take Engel's advice, and plot the logs. When we do, we see that the slope is not constant, but seems to change. One strategy is to make a piecewise function; Figure 4 (left) shows the log-transformed data with a piecewise-linear function. The right side of the figure shows the same function, back-transformed into weight and length.

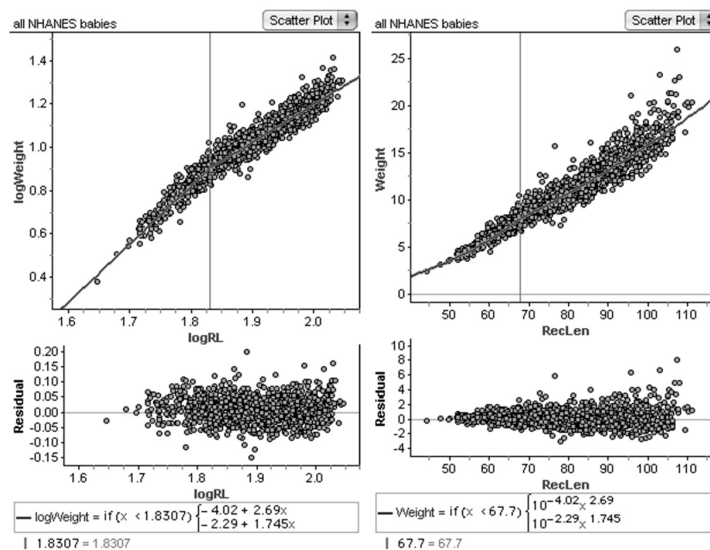


Figure 4. On the left, logarithms of weights and heights—actually “recumbent lengths”—of 1147 American children aged 0 to 48 months. The model is piecewise linear, breaking at the spot indicated by the vertical line (67.7 cm in “length”). On the right, the back-transformed, piecewise power law corresponding to that function. Heights are in centimeters, weights in kilograms.

As we suspected, the exponents are less than three—which makes sense based on our experience.<sup>3</sup> And if the change in exponents is “real,” it seems that infants (Length < 68 cm, exponent  $\sim 2.69$ ) are less dramatically chubby-when-smaller than small children (exponent  $\sim 1.745$ ).

Reflecting back on this analysis, and that of the alligators and snook, students first see that this cubic scaling property of volume works pretty well for living things such as alligators. It’s not just for nuts.

But students can go further, and discover how a non-integer exponent can fit the data even better. Furthermore, they can use logarithms to save themselves trouble (not simply to solve equations or get good grades in math). More importantly, they see that they can actually make *sense* of non-integer exponents: to describe relationships among members in a population, and to quantify the differences between the *shapes* of population members.

## 4 Hexnuts Redux

Surprisingly, we can use this insight about complicated living beings when we return to our original object of study: hexnuts.

The original hexnut data applies only to a limited range of nuts—ranging in size only from about 6 mm to 19 mm—so let’s see what we find if we extend that range. We could buy larger nuts, and weigh them. But really big hexnuts—more than 30 mm—are expensive and hard to find. Fortunately, nut manufacturers post data about their products on the Internet.

### 4.1 Not cubic after all?

If we plot data from the B&G Manufacturing website, and use all of our strategies to make a good model of the weight (or mass) as a function of the bolt size, we can get a plot like the one in Figure 5 (left). It shows small residuals, and “local” patterns that are doubtless because of rounding. The best exponent is about 2.88, and it’s clear that an exponent of 3.00 (Figure 5, right) is inappropriate.

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<sup>3</sup> Children in Medieval art were often given the same proportions as adults; this “looks wrong” to us precisely because we recognize that infants are differently shaped.

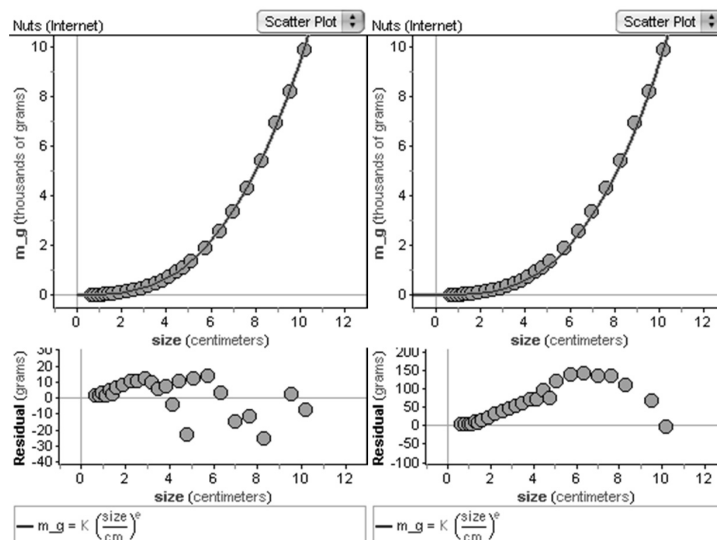


Figure 5. Modeling larger hexnuts, with data from the B&G corporation (USA). The figure on the left uses an exponent  $e = 2.88$ ; the right-hand picture uses  $e = 3.00$ . Note the scales in the residual plots—the deviations on the left are much smaller than those on the right. Our conclusion? Like snook, the nuts *are not, in fact, geometrically similar*. Are hexnuts really like babies—plumper when they are small and more sleek as they get larger? How can we explain this strange exponent in a manufactured object?

## 4.2 A better explanation—discovered through data

Further investigation of the data gives us a possible explanation. B&G supplies additional dimensions, so let us use our strategy of using data and models to gain insight. Figure 6, with least-squares linear fits, shows how the thickness (we call it *thick*) and face-to-face “diameter” (the variable *flat*) of these nuts depends on the (bolt) *size*. The two equations are:

$$thick = 0.015 + 1.00 \, size$$

$$flat = 0.32 + 1.50 \, size.$$

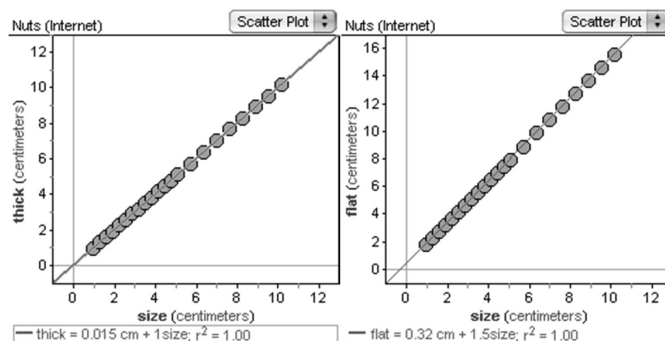


Figure 6. Nut thickness (left) and face-to-face “diameter” (right, called “flat”) as a function of nut size. Dimensions in centimeters.

These equations are remarkable and convenient: for this type of nut (these are “heavy” nuts; the ones we measured earlier are “standard” nuts) the thickness seems to be very close to the diameter of the bolt. And the distance across the nut (*flat*) is 1.5 times that diameter.

Or is it? Although the intercept for the *thick* equation is very close to zero, the one for *flat* is not. In fact, it’s almost exactly 1/8 inch. This suggests a plausible rationale for the choice of dimensions: the designer might have created a 1/16 inch thick ring—a minimum “safety zone”—around the hole in addition to the quarter-diameter thickness surrounding the hole.

If this is the designer’s original rationale, it has an interesting consequence: the weight of the nut, which depends on volume, is not proportional to the cube of the bolt size, but to a different function that incorporates this intercept.

### 4.3 The nut weight function

We will not spoil the reader’s enjoyment (or exceed our page limit) by showing this function. The author modeled a nut as a hexagonal prism with a cylindrical hole, and got excellent results.

Here is a taste, though: the reader can deduce that the function  $f(x)$  is, in fact, cubic—the product of three linear functions. But it has lower-order terms introduced by the presence of the intercept. When you plot that cubic function over the domain that corresponds to our nut sizes, you can approximate it with a  $f(x) \approx Ax^k$  power law.

When you do, the exponent is about 2.88 (just like the data)<sup>4</sup>. Outside the useful domain, the functions diverge. Figure 7 shows the *difference* between the cubic and the power law. Bear in mind that when *size* = 2.5 cm, the function values are quite large, around 190 g. So the approximation is good over most of the relevant range.

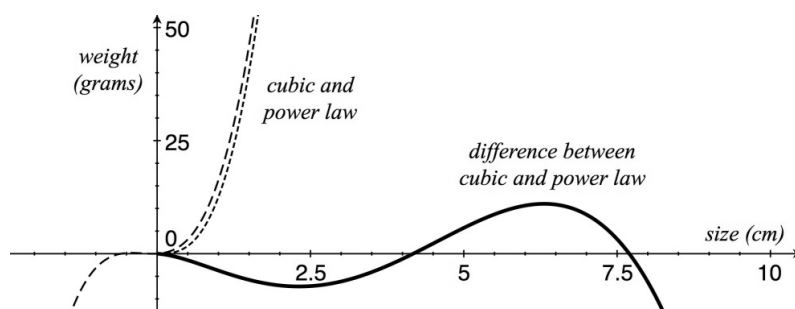


Figure 7. The model cubic function  $f(x)$  (long dashes) and the power-law approximation (short dashes) with exponent 2.88. The thick solid function is the difference between the two.

The polynomial embodies the “truth” of the relationships in this investigation. But there is nothing wrong with using a power law to approximate it. Furthermore, the exponent gives us insight into dimensionality: because the intercept adds the same amount to small and large nuts, small hexnuts are actually “chubbier” than their larger siblings.

## 5 Conclusions

It has been interesting to revisit the hexnut data, reflect on dimensionality, and extend our understanding of fastening hardware. We have explored some interesting problems we could use with our students (and pointers to the data appear at the end). But beyond that, what do we gain?

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<sup>4</sup> Isn't this an interesting twist? We are used to using polynomials (e.g., Taylor series) to approximate other functions. Here is a situation where we analyzed data using a power law, and discover that we have been using it to approximate a polynomial.

For one thing, we can use these examples to help us reflect on learning. Analysis of the skills that students need to address these tasks may be useful as a starting place for creating a more modeling-oriented curriculum.

In order to make good fits, for example, students need to know about transformations and residuals. To create geometrical models, students need to develop a feel for what simplifications are appropriate, what dimensions to measure, and how to wrestle various size relationships into a coherent equation or system.

### 5.1 The nature of modeling

Then, reflecting on the nuts, snook, and babies gives us a chance to reflect on the nature of modeling itself. Numerous authors have described modeling in various ways. Blum et al. (2007), for example, represent their modeling cycle thus:<sup>5</sup>

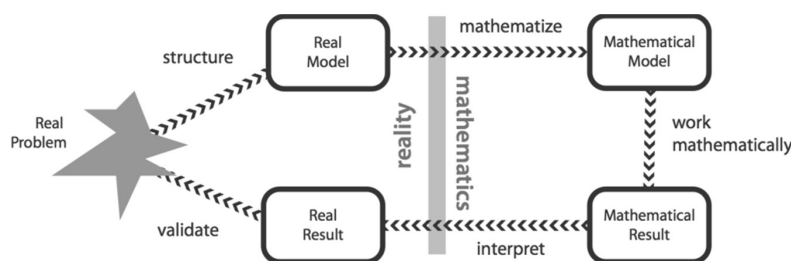


Figure 8. Author's adaptation of Blum's (2007) modeling cycle.

This is not the place to review the considerable literature on modeling. But using Blum as an example, we can ask, does this diagram really work? The test of such a proposed structure is how well it helps us understand what we do, or how well it describes what we want students to consider as they approach a problem.

<sup>5</sup> It's worth noting that any diagram like this, purporting to describe an intellectual process, is a model—in this case, a metamodel. Not a mathematical model, to be sure, but it is a useful way to think about thinking. Which, I suppose, is metathinking.

## 5.2 Hexnuts Redux as an example

Let's look at our return to the hexnuts. The analysis with the cubic function actually has two modeling episodes—two different instances of mathematizing—and they are very different from one another.

- In the first, we approximate the hexnut as a hexagonal prism with a cylindrical hole. This is a geometrical simplification: it's easier to calculate than the real volume because it ignores the threads and the easing of the corners. But we hope (and the data suggest) that the function we create captures the *essence* of how the hexnut's volume depends on size.
- The second modeling phase occurs when we plot that function on the same axes as the data, and determine what parameter value(s) make the function fit best. The function ignores small deviations, and we hope (and the data suggest) that the function captures the *essence* of the relationship between size and mass.

What distinguishes these two phases? Certainly, what the modeler does is very different. The purpose is also different: we use the first, geometrical model in order to figure out the *form* of the function; we use the second, data-and-function model to determine its parameters. Each requires a separate constellation of skills; our students need both.

What does this have to do with the modeling cycle diagram? Simply this: the diagram doesn't capture our process. To be sure, there are episodes of mathematizing and re-interpretation, but there are two of each, not just one. And what about the linear plots? In Figure 6 we plotted *face* as a function of *size*. That was modeling too—is that a *third* episode? So are they separate tasks, each with a separate cycle? Or is each one, perhaps, embedded in another?

## 5.3 Recommendations

I'm sure that any modeling theorist can explain this problem, showing how their metamodel passes the "hexnut redux" test. But at some point, I fear it may become like Ptolemy's epicycles—a lot of Byzantine explanation whose main purpose is to support the primacy of circular motion.

In practice, I suspect that our thought processes—and those we want for our students—are less cyclic and more eclectic. Different from problem to problem, and different from person to person.

The ingredients of Blum's diagram are brilliant, but we put them together in whatever order makes sense at the time. At any moment in the problem-solving process, we may need to mathematize in some way; or we may need to do purely

mathematical work; or we may need to interpret a result. It's as if the process were an arbitrary modeling-process graph—a web of nodes and edges—rather than a simple cycle.

If this is correct, it may be important for us as educators to pay more attention to the ingredients and not as much to the whole cycle. For example, we should teach students about the many ways we make something mathematical, and how to recognize when it makes sense to do so; or to help students interpret their mathematical results in real terms, and decide whether their solution works in reality.

Of course, “closing the loop”—starting with a real-world problem, modeling, iterating, and following the analysis all the way to a real-world conclusion—is still important. But the cycle diagram (like any model) simplifies reality—maybe too much. I worry that new curricula and materials might not give students the support they need to learn the individual steps, such as mathematizing, including the problem of when to do so and what kind of mathematizing to do.

#### 5.4 Final Words

It comes to this: we care about modeling in mathematics education. I conjecture (again without data) that it's personal and emotional. Modeling—numerical or geometrical, using functions or graphs, with a pencil, a piece of chalk, or with a computer—is an exhilarating mathematical activity. It's satisfying. It gives us insight. It helps us understand. It's mathematics that works.

So we want to share it.

To do that well, we must not rely on the theoreticians. Here's my suggestion: model, and pay attention. Get the data and do the hexnut problem yourself. Put on your metacognitive hat and see what skills and insights you use.

And don't stop. Interesting modeling problems are everywhere, and each one has different attributes. Perhaps in some future *Festschrift*, we will all get together and discuss what we have found.

#### References

- Blum W., & Leiß, D. (2007). How do students and teachers deal with modelling problems? In C. Haines, P. Galbraith, W. Blum, and S. Khan (Eds.), *Mathematical Modeling: education, engineering, and economics* (pp. 222–231). Chichester (UK): Horwood.
- Engel, J. (2010). *Anwendungsorientierte Mathematik: von Daten zur Funktion*. Berlin: Springer-Verlag.

## Data Resources

I have collected some of the data used in this chapter in convenient formats in the “eeps data zoo” at <http://www.eeps.com/zoo>. Some of the “exhibits” in the zoo include:

- NHANES Data prototype. This includes health data on over 10,000 Americans, including the children in Figure 4. Extracted by the author from the Centers for Disease Control in the USA. ([http://wwwn.cdc.gov/nchs/nhanes/search/nhanes03\\_04.aspx](http://wwwn.cdc.gov/nchs/nhanes/search/nhanes03_04.aspx))
- Hexnuts. The original hexnuts data, extended to include some larger—and metric—nuts.
- Heavy Hexnuts. Data from the B&G Manufacturing website (<http://www.bgmfg.com>); these are what we used in “Hexnuts Redux.”
- Coins. Fodder for your own modeling. Data on miscellaneous coins. How do you suppose diameter, thickness, and weight are related? What measurement issues arise?
- Snook. Thousands of them. From the Florida Fish and Wildlife Conservation Commission. Data via private communication, ca. 2004. These are summarized at <http://myfwc.com/research/saltwater/fish/snook/length-weight/>

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