Changing a Recurrence Relation to an Analytic Function

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Suppose you're modeling a phenomenon and you're thinking about it recursively. But you want an analytic representation. What does that mean? And how do you do it? We'll start out ridiculously simply and then make it harder.

Ridiculously Simple

Suppose you're counting the number of cockroaches entering your Roach Motel® and are trying to model the total number of roaches in the motel. If you see that two are coming in every minute, you could say

$$P_n = P_{n-1} + 2$$

That is, the total population P of roaches in any given minute (minute number n) is two more than it was in the previous minute ($n - 1$).

Now suppose you want an *analytic function* for the same thing. By "analytic function" we mean, a rule that we can use to find out the population at any given minute. It should start with "$P(n) =$" which is just like the more familiar "$f(x) =$" but with different letters.

What we have already is a *recurrence relation*, that is, an expression that uses recursion to figure out the population. As you can see, you can't just plug in n. Instead, you have to know the population in the previous minute in order to figure out the population now.

The analytic function for this situation would be

$$P(n) = P_0 + 2n.$$

Make sure this makes sense to you. P_0 is the population when $n = 0$. Notice how we have to "anchor" the population somewhere in order to find the total number of roaches; it is not enough to know that there are 2 more every minute. We have to know how many there were at minute zero to find the total.

(Note: this is just like the way mathematical induction works, or integration, or solving differential equations. Putting it another way, the recurrence relation tells you the slope, but not the intercept.)

The key thing here is that the analytic formula makes sense. Every minute there are two more roaches, and when n = 0, the population is P_0. So if there were 7 roaches in the motel at time $n = 0$, there will be 7 + (5 * 2) = 17 roaches at minute 5.

Generalizing Simplicity

We can generalize this result. If we have a recurrence relation (a.k.a. recursive function)

$$y_n = y_{n-1} + k,$$

the corresponding analytic function is

$$y(n) = y_0 + nk.$$

That is, n is the "x" variable, y_0 is the intercept, and k is the slope.

Getting Exponentially More Complicated[1]

Now suppose the roach motel situation is different: every minute the number of roached increases by 10%. We would express that recursively as:

$$P_n = 1.10 P_{n-1}.$$

That is, instead of *adding* a constant every minute, we *multiply* by that constant. So

$$P_1 = 1.10 P_0$$
$$P_2 = 1.10 P_1 = 1.10\left(1.10 P_0\right) = \left(1.10\right)^2 P_0$$
$$P_3 = 1.10 P_2 = 1.10\left(1.10^2 P_0\right) = \left(1.10\right)^3 P_0, \text{ and so forth.}$$

This situation creates exponential behavior—just like compound interest. Looking at the pattern we were coming up with, you can imagine that the analytic form is:

$$P(n) = P_0 \left(1.10\right)^n.$$

(We could prove this using induction, but let's not.)

To generalize, if we have

$$y_n = r y_{n-1},$$

the analytical version will be

$$y(n) = y_0 r^n.$$

This is one of the familiar exponential forms. If the ratio r is greater than 1, it's increasing; if r is less than one, it's decreasing.

[1] It's not a lot more complicated, honest—it just uses exponentials!

Now, Genuinely Hard

We've seen how to get analytic functions from recurrence relations in two cases: when the recurrence adds a constant, and when it multiples by a constant. But what if you do both at once? That is, suppose you have

$$y_n = ay_{n-1} + b \text{ ?}$$

(This arises in the Tinkertoy problem in *EGADs*. Find it on www.eeps.com. It also is at the root of the famous problem, *Sally McCracken and the Pig-Eyed Tricksters*. See Marilyn Burns, *Math for Smarty Pants*.)

What can we do to make this analytic? The trick is to look for patterns (as we did with the exponential) by finding a few specific terms. We expect to use y_0 in our solution (all the others have) so we'll start with y_1:

$$\begin{aligned}
y_1 &= ay_0 + b \\
y_2 &= ay_1 + b = a(ay_0 + b) + b = a^2 y_0 + (ab + b) \\
&= y_0 a^2 + b(a+1) \\
y_3 &= ay_2 + b = a\left[a^2 y_0 + (ab+b)\right] + b = a^3 y_0 + \left(a^2 b + ab + b\right) \\
&= y_0 a^3 + b\left(a^2 + a + 1\right) \\
y_4 &= ay_3 + b = y_0 a^4 + (a^3 b + a^2 b + ab + b) \\
&= y_0 a^4 + b\left(a^3 + a^2 + a + 1\right),
\end{aligned}$$

and so forth.

Do you see the pattern? In general,

$$y(n) = y_0 a^n + b\left(a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1\right), \text{ or } y(n) = y_0 a^n + b\sum_{0}^{n-1} a^k.$$

The a^n part resembles the exponential solution. But what about that summation? That's just a finite geometric series. We won't do that derivation here, but the formula is:

$$\sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a - 1}.$$

This means that the analytic form of our general linear recurrence relation is:

$$y(n) = y_0 a^n + b\frac{a^n - 1}{a - 1}.$$

Whew!

An Example

That looks so formidable, let's do an example with easy numbers:

A magical room will double the number of cantaloupes inside it each night. The zeroth night there are 10 cantaloupes. At the end of every night, after the doubling, the Cantaloupe Elf who eeps the magic running takes five cantaloupes away for her family. How many cantaloupes are there after three nights?

Answer: We could do it one night at a time.

- First night, we double 10 to get 20, subtract 5 to get 15.
- Second night, double 15 to get 30, subtract 5 to get 25.
- Third night, double 25 to get 50, subtract 5 to get 45.

Or we could use the formula we just derived.

In this case, $n = 3$, $y_0 = 10$, $a = 2$, and $b = -5$.

So we plug into the formula to get

$$\begin{aligned} y(3) &= (10)(2^3) - 5\left(\frac{2^3 - 1}{2 - 1}\right) \\ &= 80 - 5 \times 7 \\ &= 45. \end{aligned}$$

In this case, it was a tossup which was better, but if a is not an integer or n is large, the formula will save you time.

