Caging the Capybara: Understanding Functions through Modeling

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Abstract How do advances in technology expand and improve the ways we can teach about mathematical functions? In addition to indispensable tools for dealing with stochastics, technology gives us modeling tools: tools that enhance data analysis by letting students graph functions with their data, create new computed variables, and control model functions dynamically by varying the functions' parameters. In this paper, we will use an extended example—an optimization task we will call the "Capybara Problem"—to show how we can use these tools to address common difficulties students have with functions. This paper describes seven different approaches to this problem, beginning at the concrete end of the spectrum—using physical materials to represent the problem and its constraints—and then gradually introducing abstraction in the form of variables and functions. Technology supports students throughout this process, helping them understand the nature of variables, and helping them learn to construct symbolic functions and to meaning in their forms and parameters.

Introduction

Functions are among the most important concepts in mathematics—and not just for mathematicians. As lay people, we deal with functions to make observations and decisions large and small, personal and public. We often look at functions of *time*: my rent is going up. Unemployment is going down. Sometimes, we're looking at how one quantity varies with another: The faster I drive, the worse my gas mileage. The more coffee I drink in the evening, the harder time I have getting to sleep. In any case, functions are relevant because they're about *relationships*.

Traditionally, learning about functions has been an abstract and formal endeavor, rooted in algebra. Readers of this volume (especially older readers) learned about elementary functions—linear, quadratic, polynomial, power-law, exponential, logarithmic, trigonometric—from an algebraic perspective. We learned how to solve for quantities in symbolic expressions, and came to understand how combinations of symbols encoded features of functions—for example, how a positive leading coefficient in a quadratic means that the curve opens "up," or how a number added to a sine function raises the wave. Graphs helped us understand these principles, but we did not use graphs to solve problems: they were too hard to draw. We saw some data, although not very much of it. More often, we struggled with abstract symbolic representations, and learned their properties.

We can improve this. It is now easy to make graphs, and easy to find and incorporate data into problems and investigations. We should use these new tools to make learning about functions more accessible, concrete, and effective.

Developers at the postsecondary level (e.g., Engel 2010) have addressed this problem under the aegis of applied mathematics. At the school level, it is currently popular in the US to call this a *modeling* approach. Fortunately, there has recently been a greater call for more modeling in mathematics education (e.g., in the "Common Core" State standards, NGA 2010). Because of improvements in technology, it is now practical to do genuine modeling in the secondary classroom. (See also Erickson 2005 for examples.)

Let us take as given that we believe that being rooted in data is a good idea, and that graphs—created through technology—can give you more insight than a typical algebraic expression (Kaput 1989).

We will spend most of this paper dissecting an example in detail. During this journey, we will see how two ingredients—dynamic graphs and data—combine to help students make sense of functions. The graphs give us insight into the functions; the data gives us realism and rich, interesting contexts. We will also see how data, the epitome of concreteness, helps us on the road to abstraction.

The Capybara Problem

Consider this typical introductory calculus problem:

> The Queen wants you to use a total of 100 meters of fence to build a Circular pen for her pet Capybara and a Square pen for her pet Sloth. Because she prizes her pets, she wants the pet pens paved in platinum. Because she is a prudent queen, she wants you to *minimize* the total area.
> What are the dimensions of the Queen's two pet pens?

In a problem like this one, students often have trouble setting up the function to be minimized, in particular:

- Choosing a suitable *single* independent variable.
- Confusing the side length of the square with its perimeter.
- Maintaining the 100-meter constraint.
- Finding the area of the circle given its circumference.

There are some chain-rule challenges in taking the derivative, but if you can't make the original equation, no amount of differentiating will get you the right answer. In the traditional calculus class, moreover, the instructor is focused on the calculus, not on how to build the right function.

Seven approaches to the capybara problem

To see how modeling and data can help with this, let us approach this problem from different points along a "continuum of abstraction," gradually leaving behind physical materials and the construction of specific circles and squares as we gradually introduce variables, functions, and generalizations. (We can also view it as a developmental sequence.) These different approaches will help students understand different parts of the formula-creation process.

1. Each pair of students gets 100 centimeters of string. They cut the string in an arbitrary place, form one piece into a circle and the other into a square, measure the dimensions of the figures, and calculate the areas. They glue or tape the shapes to pieces of paper. The class makes a display of the shapes and their areas, organizes them—perhaps by the sizes of the squares—and draws a conclusion about the approximate dimensions of the minimum-area enclosures.
   This approach is the most concrete; even elementary students can use it. It is modeling even though it does not use functions. The experienced teacher can highlight the way the problem constrains the total amount of fence: To make a small total area, why can't you just make a little circle and a little square? Because you have to use all the string.
2. Same as above, but we plot the data on a graph. To do this, we have to decide on an independent variable—a number that tells us which of the versions of fence is which. We probably choose the side length of the square to "name" a configuration. (The dependent variable is easier; here we would use the sum of the areas.) We can estimate the dimensions and the minimum area from a sketch of a curve through the points—an informal function, if you will.
3. We enter the data into dynamic data software, plot the points, guess that they fit a parabola, and enter a quadratic in vertex form, adjusting its parameters to fit the data. This approach introduces symbolic mathematics, and needs technology in order to be practical in the classroom.

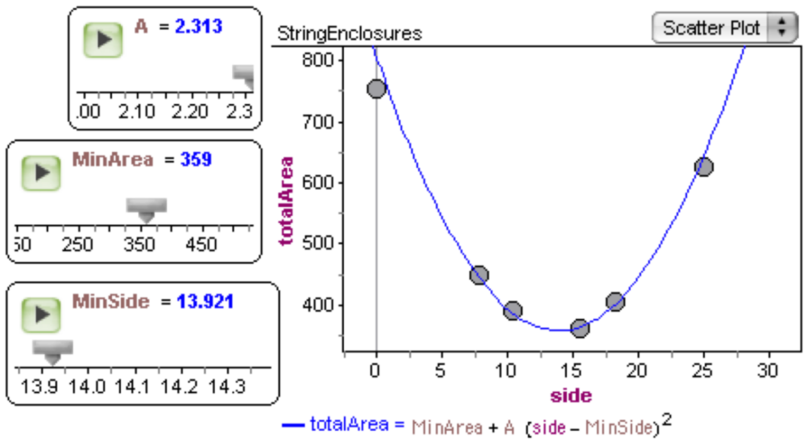

Fig. 1 Fitting a quadratic function informally using Approach (3), using Fathom

The Figure 1 shows what this looks like using Fathom (Finzer 2007); we've plotted total area (`totalArea`) against side. The sliders at left are the parameters that control the curve. These vertex-form parameters are the side of the "best" square (`MinSide`), the minimum total area (`MinArea`), and (`A`), the quadratic's leading coefficient. The function's formula here is

$$\texttt{totalArea = MinArea + A (side – MinSide)}^2.$$

You could do (1), (2), and (3) in rapid succession, to help students see connections among the string diagrams, the measurements, the graph of data points, and the graph of the function on the computer.

4. We use diagrams: instead of making the shapes with string, we draw them on paper. An individual or a small group can more easily draw several different sets of enclosures. This requires more calculation. For example, if we draw the square first, we have to calculate how big the circle must be before we draw it. This is a big step towards abstraction. The string doesn't help us keep track of the fence-length constraint, so we ourselves need to ensure that the sum of the perimeters is constant.
5. We make a diagram again, but use a variable for the length of a side and calculate expressions (instead of numbers) for all the other quantities. So we create no specific cases, we no longer measure, we no longer plot data at all. Instead, we use our expressions for the areas of the figures and plot their sum as a function of the side length—and read the minimum off the graph.

   Specifically, if the square's side length is x, its perimeter will be $4x$. The circle's circumference will have to be $(100 – 4x)$, and from that we can find the radius. We use those to find the total-area function $A(x)$:

$$A(x) = x^2 + \pi r^2 = x^2 + \pi\left(\frac{100-4x}{2\pi}\right)^2$$

This problem setup is the same as for calculus, but to find the minimum, we use graphing software and approximation. Thus this approach includes all of the abstraction that many students find so difficult.

6. We use algebraic techniques (including completing the square) to convert this expression to vertex form, from which we read the exact solution. Using this approach, we might not even plot the function.
7. We use calculus, and avoid some messy algebra in (6). Interestingly, completing the square in (6)—which students traditionally learn before calculus—is *much* harder than taking a derivative, setting it to zero, and solving.

How data, modeling, and technology help

You can see that these different approaches gradually introduce more abstraction. We conjecture that they will "scaffold" students as they work to write that A(x) function (above). Let us look in detail at the roles that data and dynamic graphs play in that process.

From data to table to scatter plot

First let us look at what data the students need to record. In the first three approaches (1–3), it is useful if they write the (measured) side of the square and the diameter of the circle; possibly the radius of the circle; the two (calculated) areas[1]; and their sum. Although the way students record data spontaneously may be informal and incomplete, this is a good chance to help them decide to organize the data into a table, with columns like a spreadsheet; this is how it will go into the computer, after all.

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[1] One could have students make the figures on grid paper and count squares to determine the area, but we'll skip that here for simplicity.

StringEnclosures

	side	diameter	S_area	C_area	totalArea
=			$side^2$	$\left(\frac{diameter}{2}\right)^2 \pi$	S_area + C_area
1	15.5	12.4	240.25	120.763	361.013
2	10.4	19.0	108.16	283.529	391.689
3	18.3	9.4	334.89	69.3978	404.288

Fig. 2 In this illustration, the students measured the `side` of the square and the diameter, but computed `S_area` (the square area), `C_area` (circle area), and `totalArea`. They did not need columns for perimeter and circumference; that will be essential in approach (4).

In approach (3), students enter this data into a table on the computer. This brings up a question: of these columns, which ones did you *measure* and which did you *compute*? And if you computed them, how did you do it? Students tend to calculate area, for example, using a calculator, and then type the result into the table cell. But instead, we can encourage them to have the computer do the calculations for the columns they compute.

For example, they probably divide the (measured) diameter by 2 to get the radius; they square that and multiply by π to get the area of the circle (see Figure 2). Asking them to write these as formulas so that the computer does the calculation accomplishes two things: it alerts students to the fact that they actually know how to perform *parts* of the calculations, and it separates what will become a very complicated formula into manageable chunks.

Adding the dynamic (and empirical) function to the graph

After students graph the data in approach (3), we introduce a dynamic function: we put a function on the graph and adjust it to fit using slider-parameters. At this level, students need to know how to write a formula for a parabola in vertex form. They need to parameterize it appropriately as well, so they can enter it into the software.

The vertex form is perfectly suited to dynamic graphing software: students see how changes in parameter values change the function. When they drag the sliders, they get a visceral feel for how the transformations work. When a residual plot is present as well, students see the parameters' effects there, for example, how changing the principal coefficient *straightens* the residuals (See Erickson 2008 for details).

Yet the symbolic form of this function will *not* look like the one that arises when we do the calculus problem—it is an *empirical* function, and therein is one of the delicate aspects of this approach. We will soon see how students come to discover the more "theoretical" form.

Having the computer calculate as much as possible

In (4), when students begin with a diagram instead of string, students have to use even more symbolic tools: they have to deal with the 100-meter constraint, and figure out the diameter of the circle given its circumference.

It helps students to have a detailed, many-columned table like the one they used above. But they now increase the number of columns, and write formulas for more of them. If they start with the side of the square, for example, they can write a formula for its perimeter. And if they know the perimeter, they can calculate the circle's circumference. They may not feel comfortable writing formulas for *all* of the derivable quantities, but that is our ultimate goal. If they start by choosing the length of the side of the square (or any particular quantity except total area), they can calculate *every* other quantity in the table—and make the graph.

Enclosures

	side	perim	circ	diameter	S_area	C_area	totalArea
=		4side	100 − perim	$\frac{circ}{\pi}$	$side^2$	$\left(\frac{diameter}{2}\right)^2 \pi$	S_area + C_area
1	10	40	60	19.0986	100	286.479	386.479
2	20	80	20	6.3662	400	31.831	431.831
3	15	60	40	12.7324	225	127.324	352.324

Fig. 3 The student is using Approach (4) in this table, computing everything based only on the side of the square. In contrast to the table in Figure 2, nothing is actually measured.

In making these formulas, students become aware of the repeated calculations they have been making; they see how to express those calculations symbolically in formulas for columns; and most important, they see that it is worth the effort to make the formulas. When they realize that they don't need to actually measure anything in the diagram to determine the two areas, they are ready to move on

Doing without the data points

Approach (5) is the modeling payoff. Instead of an empirical function that fits the data, we create the function from the geometry. The function is quadratic because it arises from area calculations, not simply because the data *look* quadratic. How does technology help students make that leap in understanding?

One way is this: in (4), once every column is computed from the side of the square, you can enter any side length you like, and the software will compute the sum of the areas. No measurement necessary. The string was a simulation of a fence; now we're using calculation to simulate the string. It is all numbers now. So we can blanket the domain with values, and plot the resulting data points, none of which depend on sliders. The points lie on a parabola, of course—the small points in Figure 4. If we could plot them all, we would have the function itself: the infinite set of points that describe the relationship.

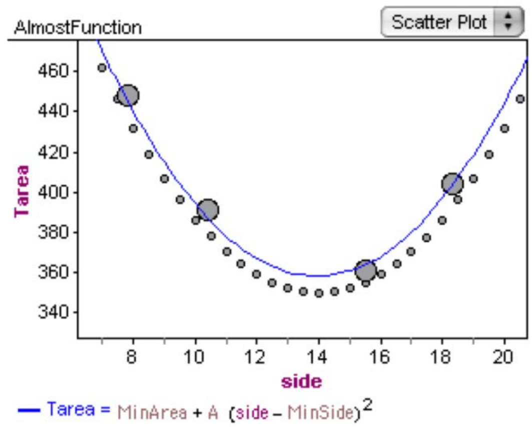

Fig. 4 In this graph, the small points are values computed from the side of the square based on a total length of 100; the function is the student's best fit to the data (the large points).

How do we turn an infinite set of points into a single curve, a single formula? We combine the formulas we have already made for each individual column. This is the spot where students need to know how to substitute—arguably a critical algebraic skill. If it is hard for them, they can do it step by step, eliminating one column at a time, and verifying that the simulation—for that is what this has become—gives the same results.

Now students can simply plot the function and find the coordinates of its minimum. They can even compare it to the vertex form; they can do it graphically, or, if they have the skills, they can put both formulas in the same algebraic form and see how well they match.

Reflecting on the process

This modeling approach addresses problems students have with creating the formula for total area. It also gives us good approximate answers to the original question—what are the dimensions of the pens?

We help students use abstraction by starting, sensibly enough, with something concrete: the string. Then we gradually introduce representations and abstractions as they become useful. This problem and these approaches are certainly not the entirety of learning about functions or about modeling in secondary mathematics, but the basic ideas should apply to other problems:

- A concrete simulation requires the least abstraction and often makes it unnecessary to model difficult-to-represent relationships.
- There are many approaches between that concrete model and the purely abstract, traditional mathematical function.

- An empirical graph—one made by measuring specific examples rather than by analyzing the generalized situation—often helps students understand underlying functions, and can lead to an approximate solution.
- An approximate solution may be good enough.
- Writing easy formulas for "bite-sized" calculations can lead to a more comprehensive solution.
- Dynamic data analysis software helps students organize their data, visualize it with graphs, create functions, and make those "bite-sized" calculations.

Finally, we should not think of using data, or graphs, or even string, as less sophisticated or desirable than using calculus. Consider one of the most confusing things about the problem—that there is in fact a *minimum* area. Most area problems using a fixed amount of fence *maximize* area. How can there be a minimum? A successful calculus student might say, "because the coefficient of the first term is positive." But a student who used the string might say, "because the shape you can make with the whole fence is much bigger than two shapes, each made with half the fence." One can make a case that the "string" answer shows more insight.

Additional Notes and Observations

Confusing the Data with the Model.
It is important that students (and teachers) be clear why the data appear as points and the model as a curve. The model is an ideal, a fantasy that we are proposing for consideration. It exists for any possible value. In contrast, data is reality, and exists only at specific values.

We can use data as a check on models (just like in science) rather than simply assuming that a formula must be better than a measurement. When we measure string figures, we will get points that do not lie on the curve. This discrepancy helps students connect math to reality. There are any numbers of reasons the curve might not go through the point. It is fair to blame the data if there is a measurement mistake (in our case, the string may have been a bit long). But the model may have missed because we modeled the ideal situation, leaving other aspects of reality *unmodeled*: our circles were not perfectly round, nor our squares square.

Habits of Mind.
We want students to develop good mathematical practices; modeling activities like this one offer opportunities. Here are two:

When we ask students to make separate columns for each quantity, and write simple formulas, we're doing two things: we're helping them encode their knowledge in chunks they understand, and learn to combine them; and we're also helping them learn to *identify and name variables*. By having them use the names of intermediate variables in formulas—instead of plugging a numerical value in, using a calculator—we're helping them see the advantage of "keeping a calculation in letters as long as possible."

Another is to *check limiting cases*. Even when students are making shapes with string, a teacher can ask, “what if we put all of the fence into the square? What would the areas be then?” Students can see that the maximum square side is 25 meters, and the sum-of-areas is 625. This point is an anchor for our curve, one we can be sure of theoretically. We can do the same on the other side, with the circle—which also reminds students how to find area if they know circumference.

Looking for New Questions.
Rich approaches like these open up new questions, often suitable for more experienced students that finish early. Here is one that we can investigate if we have a table with many columns: suppose we plot the side of the square against the diameter of the circle (see Figure 5, left). Why is it a straight line? Why is the slope –1.27?

If that is too easy, plot the area of the circle against the area of the square (Figure 5, right). What function models *that*?

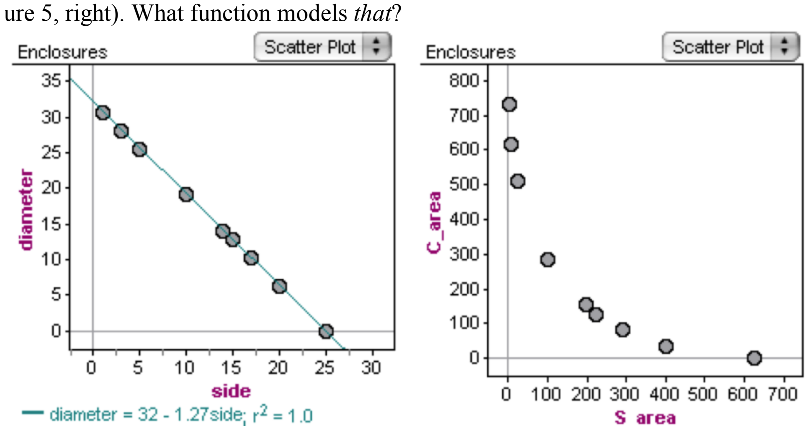

Fig. 5 Two additional relationships students can investigate

Power and insight from graphs and data—especially as things get realistic.
An abstract, symbolic solution often gives an exact answer when a model or simulation will give us only an approximation. But sometimes an approximation is what we really need.

A task like the Capybara Problem, after all, is unrealistic and idealized. It is not that a Queen would not pave her pens in platinum; but a real optimization problem would not be so clean. We would have to consider where the pens had to be placed, the slope of the ground, the cost of posts, size and cost of gates, plumbing to bring water to the area, and so forth. An idealized model, like the proverbial spherical cow, is *useful* because it captures a mathematical essence. But we can’t depend on it for the details. In our case, when we look at the graph, we can see that the curve, the parabola, is flat on the bottom, and that the total area varies no more than 10% when the side of the square is between 10 and 18 meters.

A graph tells you this vital piece of information immediately. The exact solution (*side* = 100/(4 + π)) does not.

In a modeling curriculum, we will still see these familiar and pristine problems, but we should see more and more realistic problems as well. We will see problems with more data and fewer clean and artificial constraints. We will see different kinds of functions (Thompson 1994), categorical data instead of just numerical, and data with inherent variability. Answers will become ranges instead of single numbers. We will need to cope with uncertainty in our conclusions. In short, we will need to understand stochastics and statistics.

In those problems, we will need clean, pure functions to serve as models: idealized relationships that approximate reality in some essential way. We will express them symbolically and explore their properties. But we will also have to be aware of their limitations—they are only models, after all—and use graphs, data, and all the tools of modeling to find real, practical answers and to make informed decisions.

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References

Engel, J. (2010). *Anwendungsorientierte Mathematik: Von Daten zur Funktion*. Heidelberg: Springer.

Erickson, T. (2008). A Pretty Good Fit. *The Mathematics Teacher 102*(4), 256–262.

Erickson, T. (2005). Stealing from Physics: Modeling with Mathematical Functions in Data-Rich Contexts. *Teaching Mathematics and its Applications, 25,* 23–32.

Finzer, W. (2007). *Fathom Dynamic Data™ Software* (Version 2.1). Emeryville, CA: Key Curriculum Press. Retrieved from http://www.keypress.com/fathom.

Kaput, J. J. (1989). Linking representations in the symbol systems of algebra. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (Volume 4 of Research agenda for mathematics education, pp. 167–194). Reston, VA, USA: NCTM.

National Governors Association (NGA) Center for Best Practices, Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics*. Washington D.C: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, Alan H. Schoenfeld, and Jim Kaput, (Eds.), *Research in Collegiate Mathematics Education*, *1* (Issues in Mathematics Education **4**, 21–44). American Mathematical Society, Providence, RI.