EGADs: Enriching Geometry and Algebra through Data

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Introduction

Even though many schools teach geometry and algebra in different courses, there are many situations in which they connect, largely through formulas that are about some sort of spatial phenomenon. For example, in a math book, you might see:

> A 5-meter ladder is leaning against a wall. The base of the ladder is 1 meter from the base of the wall. How high is the top of the ladder?

The point of this problem is to get you to use the Pythagorean Theorem. It demands that you combine geometry and algebra: geometry to recognize a right triangle and recall the appropriate formula, and algebra to solve that formula for the relevant side.

The activities in this booklet will give you practice making these connections. And they will do it in a special way: through *data*. You won't calculate the height of the ladder, you'll *measure* it. Here's the basic idea:

1. We present you with a situation with some geometry in it.

2. You take some measurements, record them in a data table, and plot them on a graph.

3. You figure out a mathematical function that has the same pattern as the data (this is a *mathematical model* of your data).

This last step is the critical one, of course. As you will see, there are two ways this can work:

- You use your understanding of the geometry and the situation to come up with your model.

- You use the model you find to help you understand the geometry of the situation.

In this booklet, instead of a ladder, you'll lean a chair against a wall, and measure the distance from the wall to the bottom of the chair, and from the floor to the top. Then you'll graph them and try to find the function that relates those two quantities. In a way, this is the reverse of the traditional ladder problem. Instead of going from the formula to a specific number, you'll go from numbers to the formula. That means that the point is not a particular answer, but rather a relationship.

We hope the formulas and the geometry will make more sense because they're about something real.

Real data is messy and confusing. Still, learning to handle real data is important—and it can be fun. This is partly because we will use computer software to do a lot of the graphing in this book, and that will take the drudgery out of the process.

What will be hard in this book is not the algebra, or the geometry, or even the data analysis. The hardest part will probably be connecting up the *situations* to all the math. This will often require common sense and learning to think mathematically about the situations.

<Role of prediction>

Principles

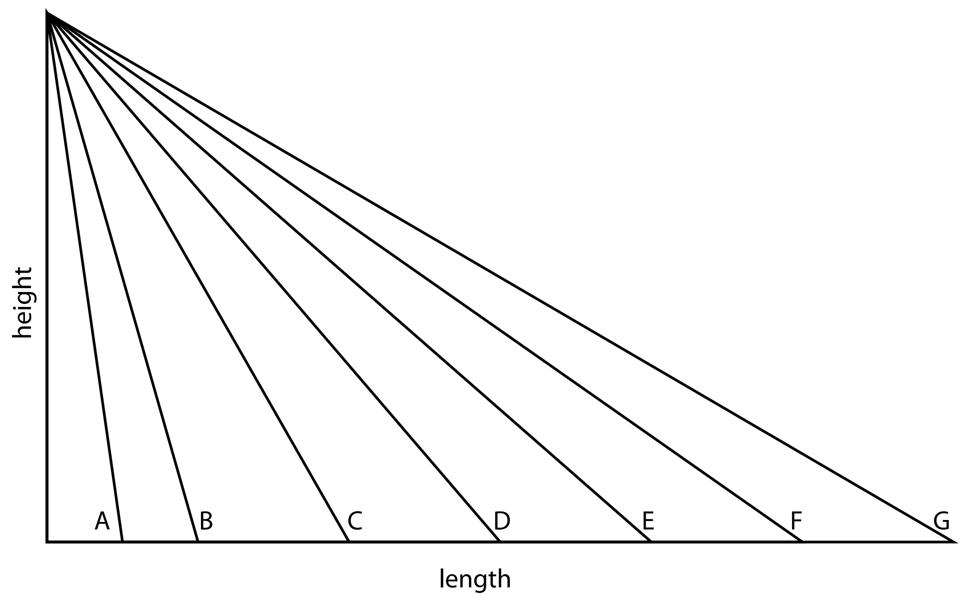
- Limiting Cases. It often helps to measure and reason about special cases, especially ones at the edges of possibility. The model you create must work properly at these special cases—and it's easy to check that out.

  So if you're doing that Pythagorean ladder problem, your formula had better work when the ladder is flat up against the wall, and when the ladder is flat on the ground.

- Residuals. When you create a mathematical model to fit data, look at the residuals. If the model is good, there will be no pattern in the residuals—they will look random—and they will be centered around zero.

1. Hypotenuse

We'll begin with a situation that calls for that famous Pythagorean Theorem.



Situation and Measurement

Let's see if the Pythagorean Theorem really holds. Does $a^2 + b^2$ really equal c^2?

The drawing has a bunch of triangles that share a common height (*a*) but different "*b*" legs and, therefore, different hypotenuses. You will measure the sides of these triangles, then see how the hypotenuse *c* depends on the changing leg *b*.

We have designed this activity with the vertical leg *a* being the same for every triangle on purpose. Nevertheless, in your data table, include *a*, *b*, and *c*.

Measure the triangles and record the lengths of their sides in the table.

triangle	height (a)	length (b)	hypotenuse (c)
A	7 cm		
B	7 cm		
C	7 cm		
D	7 cm		
E	7 cm		
F	7 cm		
G	7 cm		

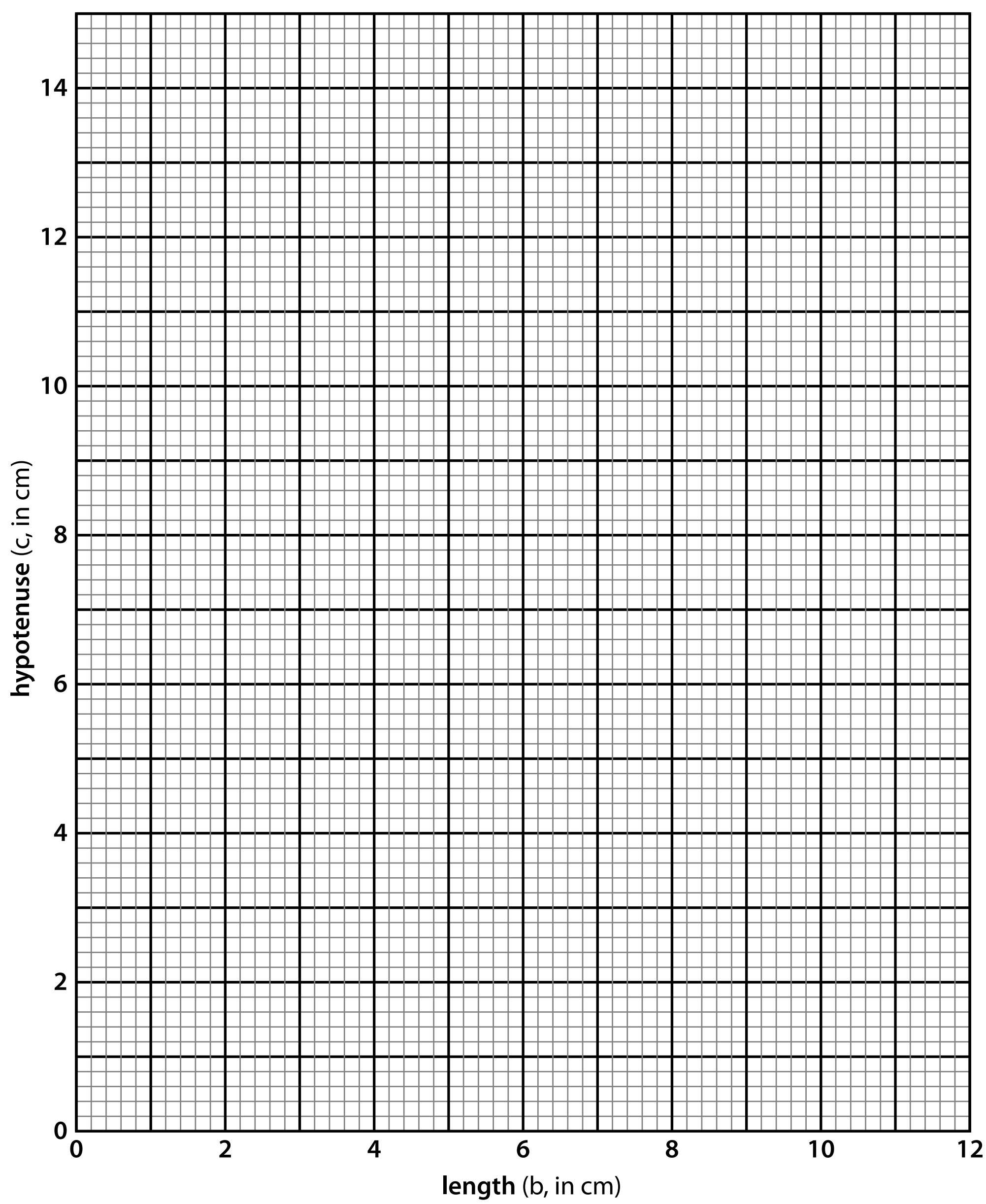
Hypotenuse Data Analysis

We will do our data analysis by hand and with a calculator in this first activity. Then we'll get the computer to help us.

Plot *c* against *b* (by hand) on the graph.

Note: when we say "*c* against *b*" that will always mean, put c on the vertical axis. We list the vertical-axis variable first. We might also say, "plot a *c*-*b* graph." It means the same thing. Here it also makes sense because *c* is the *dependent* variable: we constructed *a* and *b* first, and then connected up the endpoints to make *c*. That is, *c* depends on *b* (and *a*), not the other way around.

You have points on the graph. We want to find a curve to go through the points. You could just sketch it, but instead we want a formula so we can plug in *length* and get *hypotenuse*. Then we can draw the graph of that equation; it should go (more or less) through the points.



Hypotenuse Data Analysis, continued

In this case, we know something about the relationship between these numbers. They are, after all, sides of a right triangle. That means we can use the Pythagorean Theorem,

$$a^2 + b^2 = c^2.$$

It may not be obvious how to use this, so we'll go over it carefully right now; later on you'll need to do this on your own. Note: this is one of the big ideas in this booklet, so it's worth understanding well.

1. Figure out which quantity you want to be able to predict. In this case, it's the **hypotenuse**. Why? Because in the drawing, we set up the **length** first, and then connected up the **hypotenuse**. Notice that the thing we want to predict is on the vertical (y) axis.

2. Figure out, in the equation you have, which variable corresponds to that variable. In this case, the formula is $a^2 + b^2 = c^2$, and the **hypotenuse** variable is c.

3. Use algebra! Solve the question for the variable. In this case, it's easy: take the square root of both sides: $c = \sqrt{a^2 + b^2}$.

4. Plot that curve on your graph. In this case, you need a value of c for every value of b. You might think that there is a problem here because we have another variable: a. But we set this up so that a is the same for every triangle: 7 cm. So if we measure everything in centimeters, $a^2 = 49$, and the equation becomes $c = \sqrt{b^2 + 49}$.

   Where are x and y? If this looks confusing, you can rewrite the formula with x and y. Which variable is on the y-axis? It's c. Similarly, b is on the x-axis. So substitute to get $y = \sqrt{x^2 + 49}$.

   How do I plot the curve? One way is to make a T-table (and in-out table) of b and c values, for example:

b	c
0	7
1	sqrt(49 + 1) = 7.07
2	sqrt(49 + 4) = 7.28

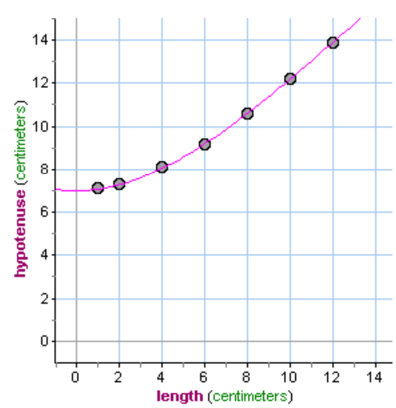
   and so forth. Plot these points and then (smoothly) connect them. Note that these points are just for helping you draw the curve. They are *not* data like the points you plotted from your measurements.

If everything worked out, the curve should mostly pretty much go through the data points.

Why does this graph matter? It shows that our measurements are consistent with the $a^2 + b^2 = c^2$ formula. It's evidence that the formula works. Put another way, if the curve did not match up with the data, there would be a serious problem, and we'd have to get to the bottom of it to make sure we understood about how triangles work.

On a practical level, having a good formula means that we can predict the **hypotenuse** for any **length**. We don't have to make the triangle—all we have to do is plug **length** into the formula. This doesn't matter a lot if you can make the triangle by drawing it on paper, but if you're building it out of steel, knowing the length of the hypotenuse in advance can save you a lot of trouble and expense.

If everything works out, your graph should look something like this:



Hypotenuse Data Analysis: Getting the Computer to Help

You will be making a lot of graphs and plotting a lot of curves. Once you know how to do it by hand, you should get a computer to help you. Here are instructions for using Fathom to make your graph:

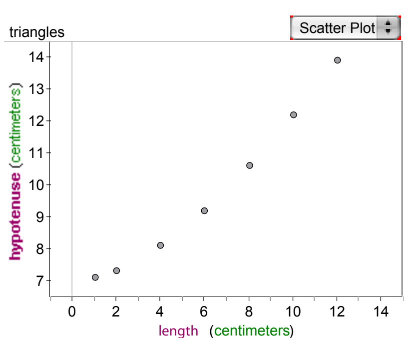
- Open up a new document in Fathom.
- Make a table by dragging one off the shelf.
- Make column headings in your table by clicking where it says **<new>**. At a minimum, you'll need columns for **length** and **hypotenuse**. You may want columns for **triangle** and **height** as well.
- Enter your data. Your table will look something like this:

triangles

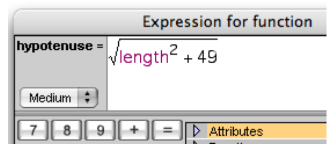
	triangle	height	length	hypotenuse	<ne
units		centimeters	centimeters	centimeters	
1	A	7 cm	1 cm	7.1 cm	
2	B	7 cm	2 cm	7.3 cm	
3	C	7 cm	4 cm	8.1 cm	
4	D	7 cm	6 cm	9.2 cm	
5	E	7 cm	8 cm	10.6 cm	
6	F	7 cm	10 cm	12.2 cm	
7	G	7 cm	12 cm	13.9 cm	

Note: In Fathom, enter units with your data.

- **Making a graph**. Drag a new graph off the shelf. It will be empty. Label your axes by dragging **length** to the horizontal axis and **hypotenuse** to the vertical axis. You'll see this:



- **Plotting a curve**. Right-click on the graph (use control-click if you have a one-button mouse) and choose **Plot Function** from the context menu that pops up (it's way down at the bottom). The formula editor appears.
- Enter the formula you want in the formula editor. Notice that **hypotenuse =** is already there. Use the keypad to get the square root symbol, and use the [^] key (or shift-6) for exponents. Your formula will look like this:



- Press **OK** to make the function plot on the graph. It may say **#Units incompatible#**. We'll fix that later.
- **Rescaling axes**. You might have noticed that the vertical axis starts around **7**. If you want to see the origin and the **length** axis, grab the number **7** by the axis and drag *up*.

A Gotcha: Where Did Height Go?

In our formula, you might wonder why we used **49** instead of **height**^{2}. Of course 49 is 7^2, and the height of all the triangles is 7 cm. But why did we put in the number instead of the expression?

Ordinarily, it's best to leave as many things as possible "in letters," that is, written using variables. But here, because we're plotting a function, *there has to be only one variable on the right-hand side of the formula*.

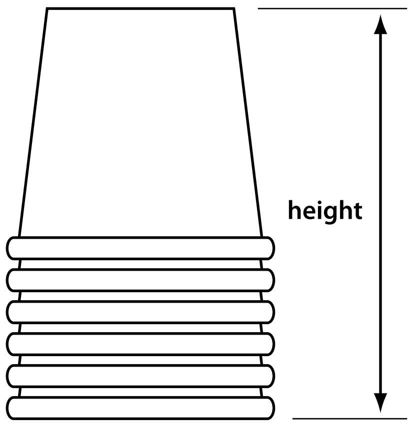
Why? To draw the curve, the computer has to calculate a **hypotenuse** for any **length**. Not just lengths in the table, but any **length** at all. If we wrote

$$\textbf{hypotenuse} = \sqrt{\textbf{length}^2 + \textbf{height}^2}$$

it couldn't calculate **hypotenuse** for **length** = 3 cm (which is not in the table) without knowing **height**. Since the height *would* be 7 cm—because of the way we set it up—we put in 49.

3. Stack of Cups

In this activity, you will make stacks of different numbers of cups. The cups "nest" inside each other. How does the height of the stack depend on the number of cups?



number	height

What to Do

Make a stack of cups, all nested together. Measure the height of the stack (as in the illustration) and count the cups (in this case, six). Record the **number** and the **height** for at least five different stacks.

How will **height** be related to **number**?

- Predict: What do you think the relationship will look like?
- Record measurements of **height** and **number**, for stacks with different numbers of cups.
- Plot **height** against **number**.
- Find a line that fits the points. Find its equation. Be sure you can explain the meaning of the slope and the intercept.
- Be sure you can explain why it makes sense that it's a line that fits the points.

Stack of Cups

We put this early in the book because it's linear, so it takes less mathematical baggage to approach. You don't need to have mastered algebra to understand this context and figure out the formula.

But it's not perfectly simple either. Even though you can use the geometry of the situation to understand the data, the meanings of the slope and intercept are subtle. Here is sample data and a graph with a good line superimposed.

<sample data>

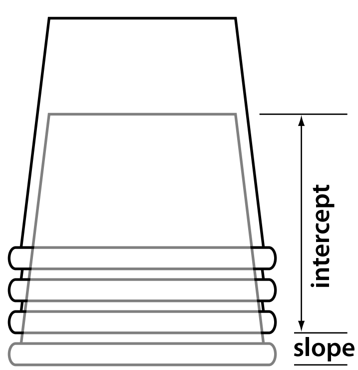
A traditional "meaning of slope" is "the height you add to a stack with each additional cup." This is correct but disconnected from the context. The intercept, however, is worse: "the height of a stack of zero cups." How could that be?

A key question is, based on your formula, *how tall is one cup*? Curiously, it's the slope *plus* the intercept. There are at least two ways to see why this is true:

- If you plug **number = 1** into your formula, you get slope plus intercept. This is a purely algebraic approach—efficient, but not showing understanding of the situation.

- If you make a diagram showing the size of slope and intercept, you can see that the size of a whole cup is their sum. This shows a good connection to the context, but is inefficient (or fragile) because "seeing" the intercept is so strange and hard.

The upshot is that we want students to have both understandings and be able to relate them.



What is really happening here is that the situation (unlike stacking without nesting) really doesn't fit cleanly with the slope-intercept form. Whereas we would ordinarily plug in zero for **number** to test our predictions—it is usually good to test limiting cases—this time zero gives us a bogus answer. Does that mean algebra is bad? No. But it does mean you can't use it without thinking.

<Point-slope alternative?><Limits of domain>

Context Note

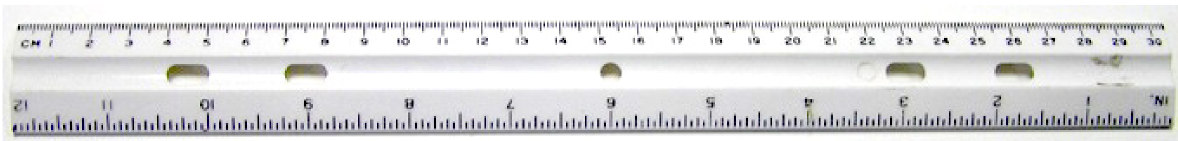
This classic situation appears frequently in math problems. Another context you will see is nested shopping carts: how does the length of a train of shopping carts depend on the number of carts in he train? Data on actual nested shopping carts appears in the eeps Data Zoo. (**http://www.eeps.com/zoo/index.html**)

4. Opposite Sides of the Ruler

In this activity, you will relate the numbers on the opposite sides of your ruler to one another.

What to Do

Take a traditional U.S. student ruler—the kind with inches on one side and centimeters on the other. For at least seven spots on the ruler, record what numbers are opposite each other.



How will **inches** be related to **centimeters**?

- Predict: What do you think the relationship will look like?

- Record at least seven measurements of **centimeters** and **inches**. For each measurement, pick a spot on the ruler and record the numbers from the two sides of the ruler.

inches	centimeters

- Plot **centimeters** (on the vertical axis) against **inches**.

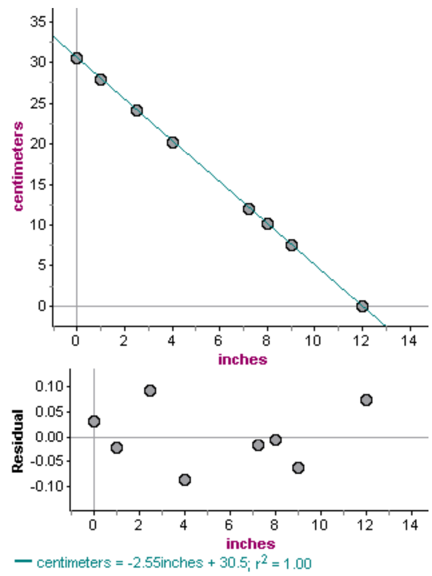
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameters.

- Be sure you can explain why the form of the function you used makes sense.

Opposite Sides of the Ruler

For most rulers of this type, the inch and centimeter scales run in opposite directions. This means that although the relationship will be linear, larger numbers on one scale will match up with smaller numbers on the other, giving a negative slope.

Here is some typical data. We have plotted a least-squares line and constructed a residual plot.



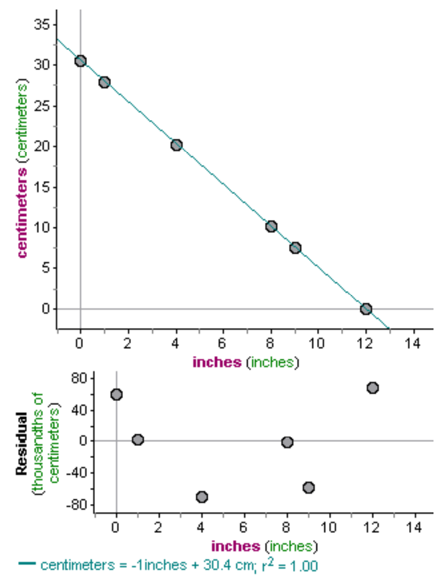
Here are some important questions for discussion:

- What goes on the y axis?
- What are the limiting cases? Why are they important?
- If you find a number outside the regular domain, such as a point with **inches = 15**, what could it mean?

The Units Issue

When we try this activity in class, some students get a slope of about –2.54, some get a slope of –1.0, and some get a slope of –0.393. What's happening?This turns out to be in interestingly deep question.

In the illustration at left, the slope value means that there are 2.55 (close to the correct value, 2.54) centimeters in one inch. But look at the next illustration:



It's the same data (minus a few points) but now the slope is –1. The data even have the same values.

How can the data have the same numbers and the computer gives you a different slope?

It's because in the second example, the student enetered the data with units. Let's compute the slope in the first example between the two endpoints, which are (0, 30.5) and (12,0): we get (0 – 30.5)/(12 – 0), or 2.54. But in the second example, the same calculation has units, so we get

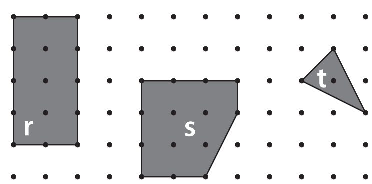
$$m = \frac{(0\text{cm} - 30.5\text{cm})}{(12\text{in} - 0\text{in})} = -\left(\frac{2.54\text{cm}}{1\text{in}}\right)$$

But 2.54 cm is the same as 1 inch, so the value of the fraction is 1.0, and the slope is –1. The first example relates the *numbers*, but the second relates the *distances*. Let that sink in!

7. Pick's Theorem

In elementary school, you may have played with cool manipulatives called geoboards. There was probably a grid of plastic poles, and you would stretch rubber bands around them to make shapes. These shapes are polygons, by the way: they have straight edges.

The key concept behind many geoboard activities is area. In order to find the area of some odd shape, you would not use formulas, but rather ingenuity. Look at the three shapes below:



The area of shape **r**, a rectangle, is 8.

What about **s**? Think of it as a square with a corner cut out. The square has an area of 9. The corner is a triangle; if you took two of those (and rotated one a half-turn) you could put them together into a 1-by-2 rectangle. That means the triangle has an area of 1, so shape **s** is (9 – 1) or 8 square units.

Shape **t** is like **s**, but harder to see. It's a 2-by-2 square (area 4) but with three triangles cut out—two "1"s and the small one, with area ½. So the total cut out is 2 ½, which leaves 1 ½ in the triangle.

Doing the Pick

This gets harder as you get stranger and stranger shapes. Georg Alexander Pick found an elegant way to figure out these areas, and you're about to find it as well. Here is what you do:

Use the grid on the handout to make a bunch of polygons. All of their vertices must be on the dots.

- For each polygon, find the area **A**.
- Also, count the number of boundary points. That's **B**. For polygon **s**, **B** = 10.
- Then count the number of interior (contained) points. That's **C**. For polygon **s**, **C** = 4.
- Assemble all this data in a table.
- Use your data to figure out how to calculate **A** using **B** and **C**.

This means you're going to make a scatter plot with **A** on the vertical axis. But what do you put on the horizontal?

Controlling Variables

One good strategy is to *control variables*. That means looking at all of your predictor variables (a.k.a. independent variables)—in this case, **B** and **C**—and holding one constant while you vary the other.

For example, you might look at only your polygons with **B** = 3, like triangle **t**. (These will all be triangles, but of different shapes and sizes.) Then you could see how the area **A** depends on the number of interior points **C**.

Then hold **C** constant: make shapes with the same number of interior points but with different numbers of boundary points (**B**). Then see how **A** depends on **B**.

Gradually find a formula for **A** that has both **B** and **C** in it, and works for all cases.

Using Fathom to Help

Fathom can help you with this. If you have a bunch of data for all different values of **B** and **C**, but you want to see how **A** depends on **C**, do this:

- Make a scatter plot with **A** on the vertical axis and **C** on the horizontal.
- Drag **B** into the middle of the plot, holding down the shift key before you drop it.
- The plot will now indicate the different values of **B** in different symbols.

Pick's Theorem

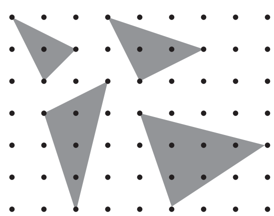
This is for mathematically more mature students.

As written, the activity gives students very little scaffolding. This is intentional. It would take a lot of paper to write a comprehensive set of instructions for how to control variables—and who would read it? This is the sort of thing you have to experience, and wrestle with, to understand.

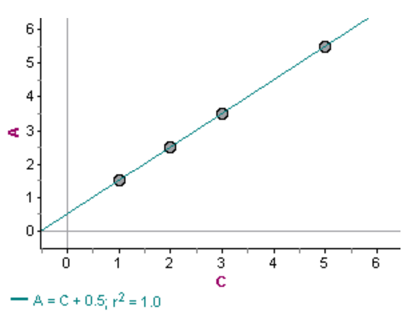
Then the idea of puttiing the general formula together is probably different from the sort of thing that students have done before.

Nevertheless, using data and graphing, we think that this experience is more accessible to more students than before.

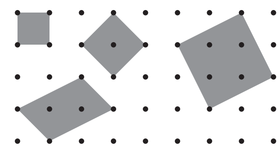
Here, for example, are 4 triangles with **B** = 3. The areas are 1 ½, 2 ½, 3 ½, and 5 ½, with 1, 2, 3, and 5 interior points. I smell a pattern…



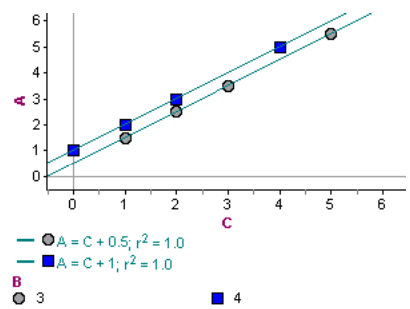
If we plot **A** against **C**, the number of interior points, and add a least-squares line[1], we see this:



Having gotten this far, you can do as the student page suggests and control the number of interior points, or you could control for boundary points again, but with 4 points instead of 3:



If you use the Fathom strategy on the student sheet, this yields:



Note that we can already predict. For example, it looks from the graph as if a polygon with **B** = 3 (marked by a circle) and **C** = 0 should have an area **A** of ½. and that's exactly right.

Oh: you want the answer? We won't spoil it for you here, but you can find it easily on the web. They often use **E** for edge or **I** for interior.

Note that this is different from other EGADs activities because there is no measurement error (you would never get an area of 11.94, or 3.2 points inside the polygon). Therefore all points should fit the model exactly.

1 We could use a movable line instead, but in this case, a least-squares line saves us some effort.

8. Cornbread Are Square

In this activity, you will explore the areas of circles.

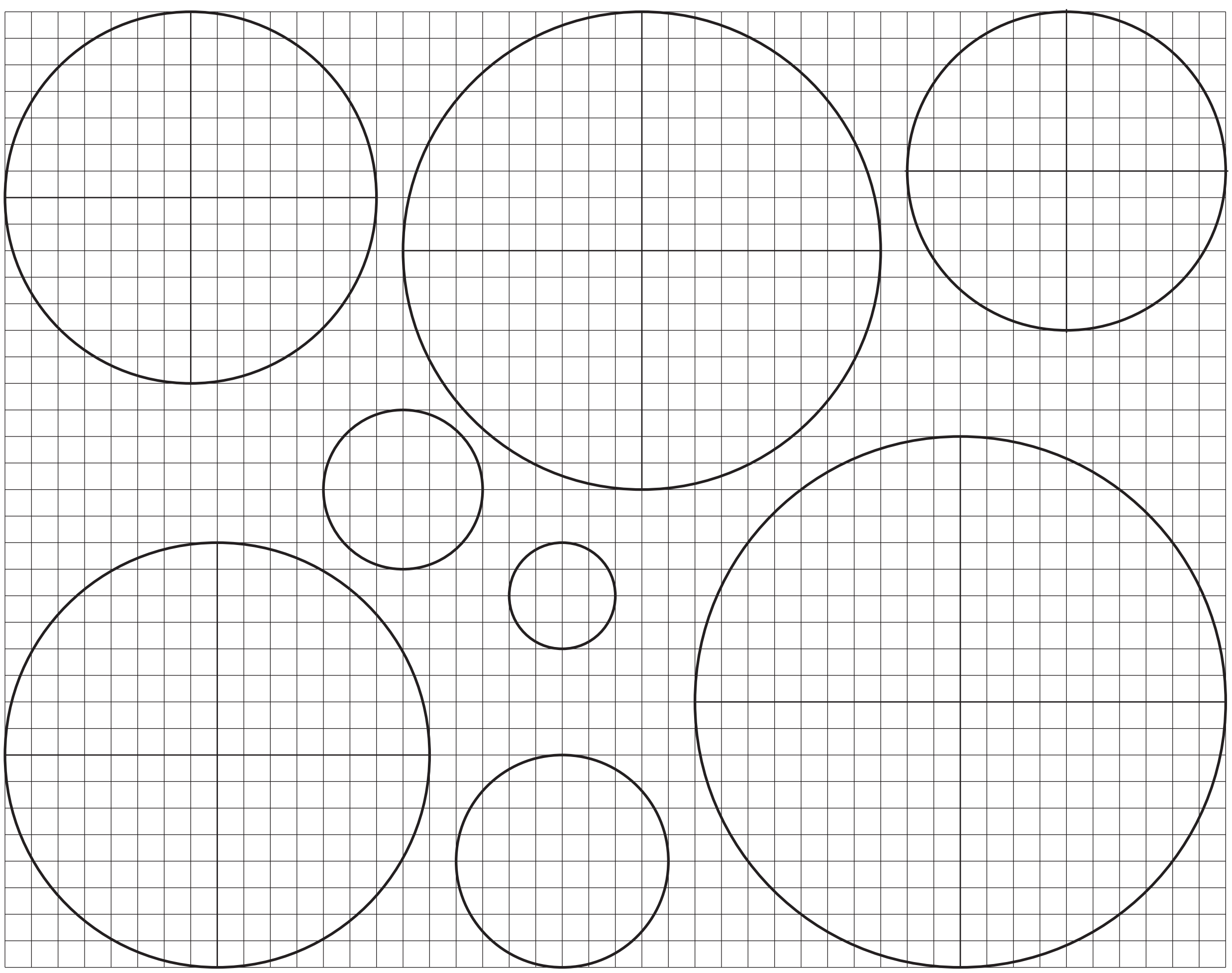
What to Do

Count the squares in each of the circles below (or you could make your own using graph paper). Count the area in the partial circles as well, estimating how many whole squares they make up together.

You will measure area and radius in the units of the grid.

How will **area** be related to **radius**?

- Predict: What do you think the relationship will look like? If you can, be precise and quantitative.
- Record measurements of **area** and **radius** for as many circles as you can.
- Plot **area** against **radius**.
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.



Cornbread Are Square

"Pie are round; cornbread are square"—Anonymous

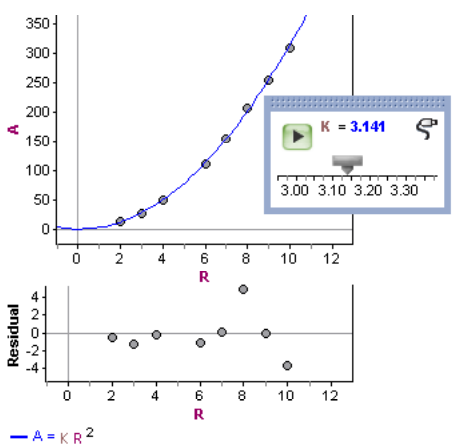
Even if students can recite the formula $A = \pi r^2$ in their sleep, the data approach is interesting, because

- They may not have thought about what area actually means in a while;
- They may never have had this formula confirmed for themselves; and
- Students may not even think of this formula in this context.

We expect students to count the squares to get the area. They could count a quarter of the squares in each circle, then multiply by 4. That's fine.

The most common way to count the partial squares is to estimate which partials go together to make up whole squares, and shade them in as you count.

Here is some pretty good data. The residual plot is helpful and illuminating; students will see that the larger circles have more leverage in the residual plot.



Why Use Functions?

You could just calculate "pi" for each circle, taking the area and dividing by the square of the radius. Why go to the trouble of plotting the function?

For one thing, it shows us the relationship, not just the number.

for another, the numbers get more accurate with larger radii; if we simply averaged the πs, we would get a skewed result.

Note: you could plot the area against the square of the radius; then you should get a straight line that (limiting cases) passes through zero, and has a slope about π.

Alternative Approach

Instead of counting up partial squares, you could have students "bracket" π by doing an upper/lower limit dance:

- Count the squares that are *entirely* within the circle. This "area" is the minimum area.
- Next, count the squares that contain any part of the interior of the circle. (That is, the interior squares pus all squares the circle goes through.) This is the maximum area.
- Make one graph of the minima, one of the maxima, and find the parameter that corresponds to each one.

  (You can also put both on the same graph; be sure to drop the second variable onto the plus-sign as shown on page ppp.)
- Notice how the larger circles push the parameters closer together.

9. Cardboard Squares

In this activity, we make rough squares out of cardboard and weigh them. How does the mass of these squares depend on their size?

What to Do

Cut squares of different sizes out of cardboard. Do not measure ahead of time. Just try to make them square. Make the squares of different sizes, as wide a range as you can with your materials. For each square, weigh the cardboard square (to get **mass**) and measure the length of a side (call it **side**). Because the squares are not perfect, some sides will be longer than others. Just pick one.

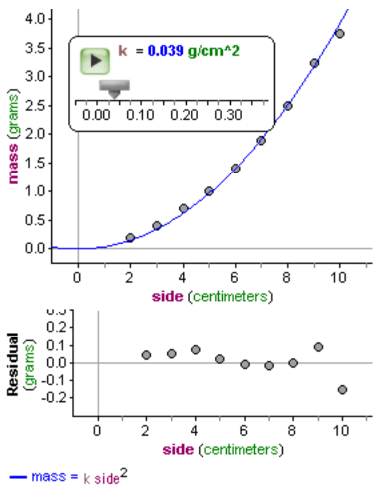
How will **side** related to **mass**?

- Predict: What do you think the relationship will look like?
- Cut out cardboard squares ranging from small (about 1 cm) to as large as possible. You must have at least 6 squares.
- Record measurements of **side** and **mass** for each of your squares.
- Plot **mass** against **side**.
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.

Cardboard Squares

If the cardboard is uniform, the mass must be proportional to the *area* of the square, or proportional to the *square* of the side. Your model will be quadratic.

Here is an example of a completed graph, with residuals. Remember: to plot a function, choose **Plot Function** from the graph's context menu.



The Fathom slider **K** serves as a parameter for the model function (see it in the function at the bottom?). The slider has units. This is the best way to get that pesky **#Units incompatible#** error to go away. If you put the units in the slider and not in the function, you may get a better idea what the parameter means. In this case, the units need to be *grams per square centimeter*, which suggests—correctly—a kind of area density.

Materials Notes

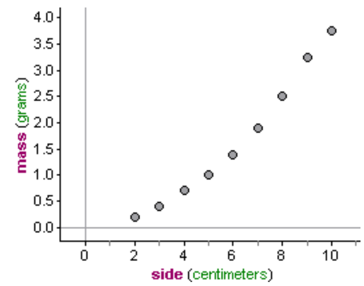
For this activity, students need cardboard, scissors, rulers, and some sort of scale.

About the scale: students need to weigh small pieces of cardboard. For some cardboard (as you can see from **K**!) it takes several square centimeters to make even a tenth of a gram.

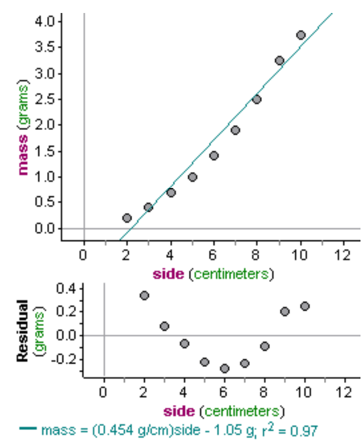
Triple-beam balances can do 0.1g, but they can be hard to use. Unless it's important that students know how to use these, go electronic at 0.1g resolution (0.01 g is better). Still, it can be hard to get electronic balances. Get a couple and share; or try to borrow them from Chemistry.

Why It's Not Linear

If you look at the data without the parabola (and without thinking too hard), you might think the relationship is linear:



But there are several good ways you can tell it's not linear even without fully understanding the problem. Here is a graph of the best-fit line with a residual plot. The value of r^2 is 0.97. What's not to like?

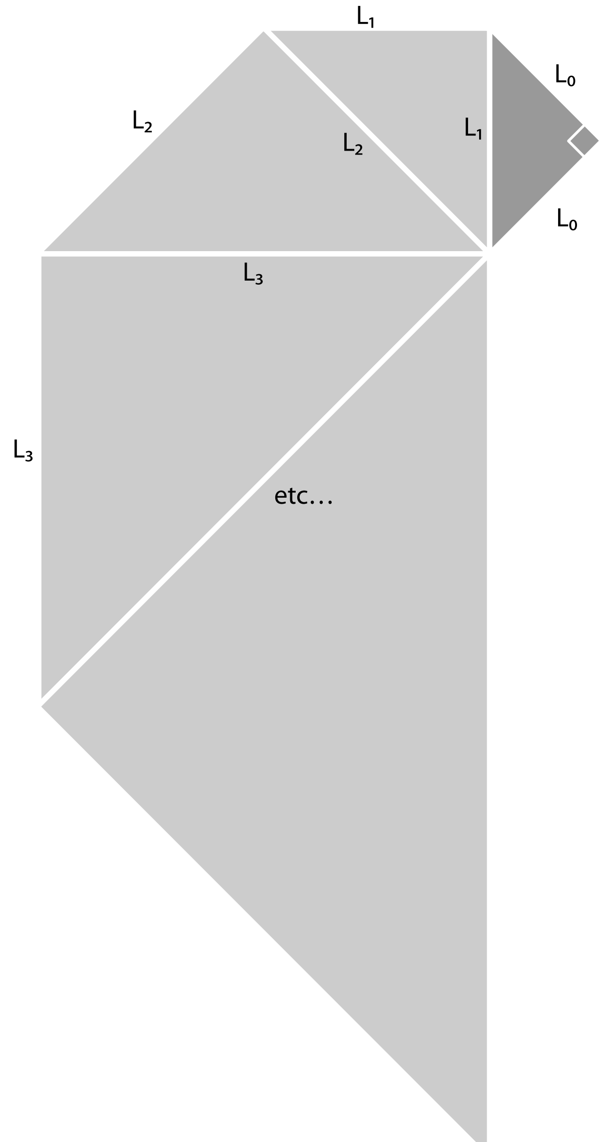


- Limiting case: The line predicts that if the side is zero, the mass will be –1.05 grams. This is ridiculous; it doesn't make sense. The mass should be zero. And it's too far off zero to be measurement error.
- Residuals: The residual plot shows a marked "bow" pattern. It's not random.

Looking at the limiting case requires that you think about the context, so is the most powerful. But if you miss it, the residuals will tell you that you neglected something important.

14. Spiral 45

In this activity, you will explore the lengths of sequential radial segments in a "triangle spiral" like the one at right. All of the triangles will be similar: isosceles right triangles in fact.



What to Do

First you have to make one. You could use the illustration here, but it's more fun—and you will get better results—if you carefully make and measure a larger drawing.

Here's the plan:

- Make a line segment, $\mathbf{L_0}$.
- Make another segment the same length as $\mathbf{L_0}$, at right angles to your original $\mathbf{L_0}$ segment and connected at one end.
- Connect these with a hypotenuse. (This makes an isosceles right triangle.)
- Call this hypotenuse $\mathbf{L_1}$.
- Make another segment, the same length $\mathbf{L_1}$, perpendicular to the first $\mathbf{L_1}$ segment, again connected at the far end.
- Connect the end to the vertex again, calling this new hypotenuse $\mathbf{L_2}$.
- Keep doing this as long as you can.

How will these $\mathbf{L_N}$s (you can name the Fathom column **L**) be related to **N**, the number of the segment?

- Predict: What do you think the relationship will look like?
- Measure **L** and record **L** and **N**. Be sure to *measure* $\mathbf{L_N}$—don't calculate it!
- Plot **L** against **N**.
- Find and explain a mathematical function that fits the points as well as possible. Be sure you can explain the meaning of any parameter.
- Explain why the *form* of the function you used makes sense.

Spiral 45 Instructor Notes

If students have a lot of experience with 45-45-90 triangles, they may naturally recognize that the hypotenuse is always $\sqrt{2}$ times the base, so the lengths of these segments will be

$$L_n = L_0\left(\sqrt{2}\right)^n,$$

which is correct.

Of course, most students won't have that epiphany quite so fast. What are other routes to understanding?

First, students may notice that if you skip a triangle, the numbers roughly double, that is, $\mathbf{L_3}$ is about twice $\mathbf{L_1}$, $\mathbf{L_4}$ is about twice $\mathbf{L_2}$, and so forth. And for some students, this may be enough.

If they have any experience with exponential functions, however, you can take that observation and help them find, for example, a formula for the even-numbered lengths.

If students think it just *looks* exponential, they could use a standard formula such as

$$L_n = Ar^n$$

and use sliders to figure out **A** and **r**.

With a little more experience, you can help students reason that since the triangles are similar, the ratio of hypotenuse-to-leg cascades upwards around the spiral. If we call that ratio **r**, we see that $\mathbf{L_1} = \mathbf{H_0} = \mathbf{rL_0}$; $\mathbf{L_2} = \mathbf{H_1} = \mathbf{rL_1} = \mathbf{r}^2\mathbf{L_0}$, and so forth.

If students do not see that **r** must be $\sqrt{2}$, they can model the data with Fathom, making **r** a slider. When they get **r** ~ 1.41, you can now ask what the significance of that number is.

Should the Initial Leg be a Slider?

You might think that since the length of the initial leg ($\mathbf{L_0}$) is really a constant, you should just type in the number instead of making it into a slider parameter.

It turns out to be better to make it a parameter just like the ratio **r**. Like so much of our data, it's a measurement, and you can never measure it perfectly. You might even be able to get a better estimate for that length looking at the whole relationship between **L** and **N** than you can by simply measuring.

An alternative is to use the first measurement as that parameter instead of typing in the number or making a slider. To do that, use the Fathom formula **first(L)**, that is,

L = first(L) * sqrt(2)^N

should be a pretty good model—but then you don't get to tweak any parameters if it doesn't fit!

Why Start at Zero?

That is, why s our first leg $\mathbf{L_0}$ instead of $\mathbf{L_1}$?

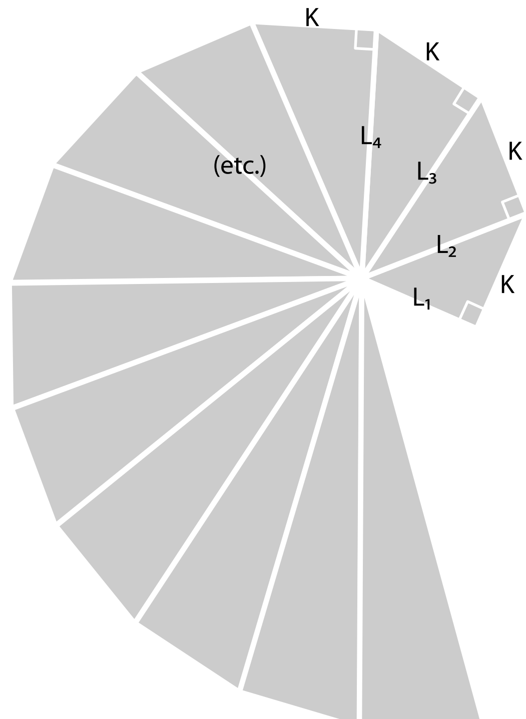
This is a good question for experienced students. The answer is, it's for convenience, so that when **n** is in the exponent, the power term is "1" the first time through. In contrast, if you started at 1, the obvious answer would have to be

$$L_n = L_1\left(\sqrt{2}\right)^{n-1},$$

which is a less elegant and makes you wonder where the –1 came from in the exponent.

15. Triangle Spiral

In this activity, you will explore the lengths of sequential radial segments in a “triangle spiral” like the one at right. This is kind of like Spiral 45 (page ppp) except that this time the triangles are not similar.



What to Do

Make your own spiral. Again, you could use the illustration here, but you will get better results with a bigger drawing.

Here’s the plan:

- Make a line segment, $\mathbf{L_1}$.
- Make segment **K**, the same length as $\mathbf{L_1}$, at right angles to $\mathbf{L_1}$ and connected at one end.
- Connect these with a hypotenuse. (This makes an isosceles right triangle.)
- Call this hypotenuse $\mathbf{L_2}$.
- Make another segment, the same length **K**, perpendicular to $\mathbf{L_2}$ at the “far” end.
- Connect the end to the vertex again, calling ithis new hypotenuse $\mathbf{L_3}$.
- Keep doing this as long as you can.

How will the length $\mathbf{L_N}$ (call it **L**) be related to the number of the segment **N**?

- Predict: What do you think the relationship will look like?
- Measure **L** and record **L** and **N**.
- Plot **L** against **N**.
- Find and explain a mathematical model that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.

Note: you may want to create a *recursive* formula for a predicted value for **L** (**Lpred**, say) instead of an *analytic* formula. Plot it on the same graph as **L**. If you want residuals, calculate them yourself.

A recursive formula for **Lpred** will use the previous value of **Lpred** in the formula instead of **N**. In Fathom, you can use the function **prev(Lpred)** to get the previous value.

Triangle Spiral

In this situation, it will be hard for students to come up with a quantitative prediction without actually calculating it all out. (They can still make good qualitative predictions, however, so don't let them off the hook.)

This is therefore a candidate for measuring first and then looking, intelligently, for a model.

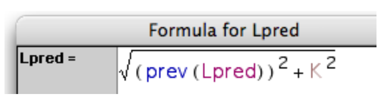
Recursive Approach

This is a good time to make a *recursive* rather than an analytical model. Traditional math classes emphasize the analytical, but the real world recurses a lot.

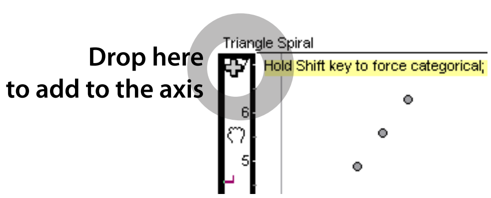
A recursive sequence uses previous values of the sequence to compute subsequent values (as in the Fibonacci sequence). In our case, we can calculate each element using the Pythagorean Theorem, like this:

$$L_n = \sqrt{L_{n-1}^2 + K^2}$$

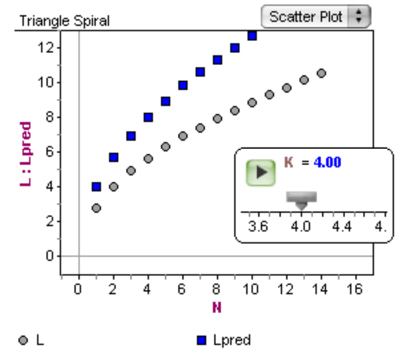
In Fathom, as the student page suggests, make a slider **K** for the constant length, and a new column, **Lpred** (for predicted length). Give it a formula:



Then put **Lpred** on the same axis as **L** in a graph. To do this, drop **Lpred** on the plus sign that appears when the cursor is over the **L** axis.



Your graph will look something like the next illustration. You can see that the parameter **K** is too large; vary it until the points line up as well as possible.



Analytical Approach

Instead of making a set of points, is there some function we can plot as we ordinarily do? Here are several strategies for finding the function:

- Just look, and try different forms, and see what works. If you get something the right shape, change the function to include parameters to stretch it.

- Do a Pólya: figure it all out with simple numbers. If **K** is 1, the $\mathbf{L_2}$ is $\sqrt{2}$. If that's true, then $\mathbf{L_3}$ is

  $$\sqrt{(\sqrt{2})^2 + 1} = \sqrt{3}$$

  Then we find that $\mathbf{L_4}$ is $\sqrt{4}$, $\mathbf{L_5}$ is $\sqrt{5}$, and so forth. This suggests that $L(n) \propto \sqrt{n}$, which is correct.

- To be really mathy, we square both sides of our recurrence for L_n:

  $$L_n^2 = L_{n-1}^2 + K^2$$

  Then we substitute $A_n = L_n^2$. This means that

  $$A_n = A_{n-1} + K^2.$$

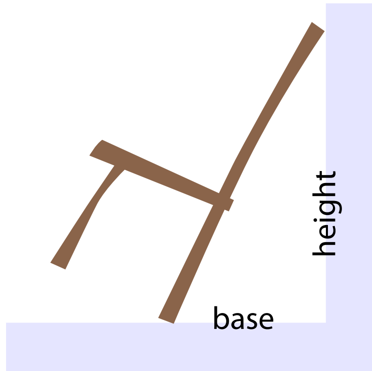
  Since $A_1 = K^2$, it follows that $A_n = nK^2$, or

  $$L_n = K\sqrt{n}.$$

Note: Even though this continuous function *works*, it doesn't make perfect *sense* for non-integers: you can never have two and a half triangles in your spiral.

18. Tilting Chairs

In this activity, we tilt a chair against a wall and see how the distances vary.



What to Do

Tilt a chair against a wall. The chair will touch the wall at some particular point. Measure the distance from that point to the floor (call it **height**) and the distance from the wall to the legs of the chair (**base**). (See the illustration.)

How will **base** be related to **height**?

- Predict: What do you think the relationship will look like?
- Record measurements of **base** and **height**, tilting the chair different amounts between measurements.
- Plot **height** against **base**.
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.

Tilting Chairs

Many students (and teachers) have a surprisingly hard time with this activity. If they do not immediately see the Pythagorean nature of the situation, this is a great opportunity to reason without Pythagoras.

Ask, for example, "when the chair is close to straight up, and you start to move the bottom away from the wall, what changes faster, the base or the height?" (the base)

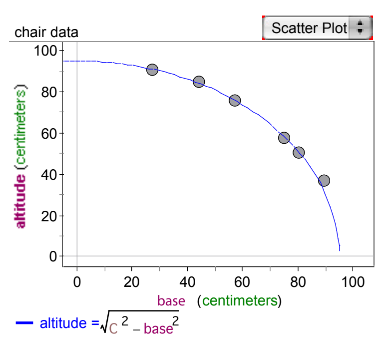
"What does that mean the graph looks like there?" (It is close to horizontal, high on the *y*- that is *height*-axis.)

Note that on the graph you get, the shape does not reflect the shape that you see in the chair even though it's spatial and arranges well. The data point is a point in space, not attached to the chair.

Point out the importance of limiting cases. Who measured the chair on the floor?

Results

Here is a graph of some typical data, with the function superimposed:



19. Vegetable Matter

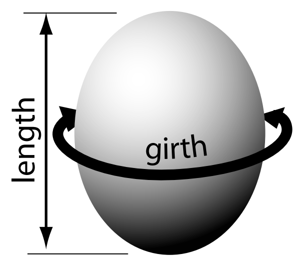
In this activity, we measure a variety of fruits and vegetables, and analyze relationships among out measurements.

What to Do

Get a collection of fruits and vegetables of diverse size. Many will be roughly spherical and more or less solid, but include some that are oblong or hollow or otherwise unusual. Bananas, sweet peppers, and cabbage work well for this.

- Measure each vegetable (or fruit; we'll just call them vegetables from now on). Measure **length** and **girth** with a tape, and **mass** using some kind of balance or scale.

Note: length is the straight distance pole-to-pole, that is, from the stem to the opposite end, measured straight. Girth is the distance around the "equator." See the diagram.



- Since you have three quantities, explore the data with any number of plots.
- Try to find relationships among the three variables and mathematical functions to act as models. The data will have a great deal of variation. The point is not to fit all the vegetables exactly, but to make a model that represents as many of them as possible, fairly well.

  You also may find functions that act as *limits* to your data.
- Be able to explain why some vegetables do not fit your models.
- Make a new variable, **predMass**, that is the predicted weight; it should be based on **length** and **girth** alone, and should work reasonably well for more-or-less solid vegetables.
- Be able to explain why the formula for **predMass** makes sense, and explain the meaning of any parameter.

Vegetable Matter

This activity is fun because of the vegetables and challenging because vegetables are so inexact.

Materials

You need fruits and vegetables, of course. Students can share. They will also need tape measures (or rulers and string) and some kind of balance or scale. Electronic balances work great.

Comments

Students may not know the formula for the volume of an ovoid; give it to them if they seem to need it:

$$V = \frac{4}{3}\pi abc,$$

where *a*, *b*, and *c* are the three "radii," the semi-major axes of the ovoid.

They also may never have realized that "solid" vegetables probably have a density close to 1.00 g/cm^3. This may be good to discover.

More importantly, they may never have used circumference (girth) to find radius or diameter, as this is backwards from most math problems. This activity ultimately asks them to put together several common formulas into an uncommon one of their own devising.

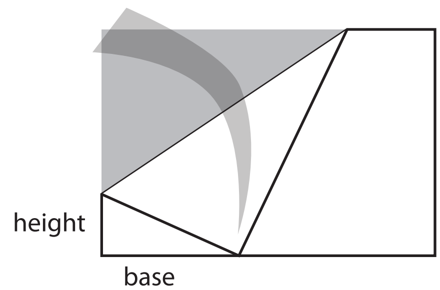
20. Triangle Folding

In this activity, you will make triangles by folding a piece of paper, and then relate the area of the triangle to other measurements.

What to Do

Take a piece of paper and orient it sideways.

Take the upper-left corner and fold it down to some point on the bottom edge of the rectangle. You have now made a triangle (See the illustration).



Measure **base**, the "bottom" side of the triangle.

Measure **height**, the "left" side of the triangle.

Calculate the **area** of the triangle.

How will **area** be related to **base**?

- Predict: What do you think the relationship will look like?
- Record measurements of **height** and **base**, folding the paper to different points.
- Calculate **area** for each triangle. Don't do this by hand! Have the computer do it:

  Make a **<new>** column called **area**.

  Right-click on the heading, area, and choose **Edit Formula** (use control-click if you have a one-button mouse)

  In the formula editor, enter the formula you want.
- Plot **area** against **base**.
- Where do you fold the paper to get maximum area for the triangle?
- How do you know there has to be a maximum?
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.

Triangle Folding

If you have never seen this one before, you're in for a treat.

If you're an experienced math-and-measurement person, you might say to yourself, "We're plotting an *area* as a function of the *length* of a side. Area always goes as side squared, so it's some kind of quadratic."

You may even go further and say, "Look: if you fold it directly down, you get zero area. And if you fold it far enough across—in fact, so that base equals the width of the paper—you get zero area. So there is a maximum in the middle. It's a parabola with two zeroes, so it's of the form

area = K * base (base – W),

where **K** is a parameter and **W** is the width[1] (the short side) of the paper."

You would be wrong.

How to Get It Right

There are several approaches that work. One is simply to figure it out algebraically. For that, you need to see that height plus the hypotenuse is the width of the paper.

But suppose you can't figure that out. Suppose you have the insight that the area has to be zero at **base = 0** and **base = W**. You try that parabola and find it wanting.

So on a whim, or perhaps with some intuition (for example that there is no way you can see an exponential coming in here or a trig function, so maybe it's still a polynomial, but of higher degree) you say, maybe it's cubic.

If that's the case, there will be another zero, and you can write:

area = K * base (base – W)(base – P)

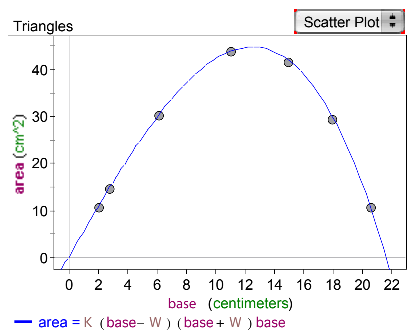
where **P** is another parameter, the location of the new zero. (**K** is also a parameter, **W** is a parameter but we know its value, and **base** is our variable.)

Playing with the parameters to fit the data, you discover that **P** is very close to **–W**.

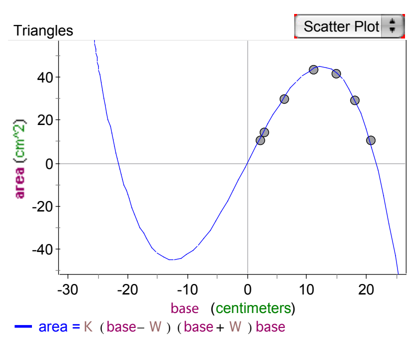
Does this make any sense at all? Surprisingly, yes. If you imagine extending the base of the paper to the left in our diagram, you could fold the corner to that line, but the triangle you get would be on the *back side* of the paper. It's almost as if—dare we say it?—it has *negative area*. Then, as the point you fold the corner to gets farther and farther out, it eventually goes to zero when the distance (**base**) is again equal to **Width**, albeit on the negative side.

Results

Here is a graph of some good data:



And if you zoom out, you can see the other side of the function:



The parameter K is about –.0116 cm^{-1}. What does it mean? For that you need to really solve the problem. It's (**–1/(4W)**).

1 Note: if you are using Fathom, do not use **Width** as the name of your parameter. That's the width of your case icons.

23. Making a Cone

In this activity, you will construct and measure cones, and relate various measurements to one another.

What to Do

See the cone template on page 55ppp. You can make a cone by shaping and connecting any sector of a circle. The template will help you make many different cones all from a circle of the same radius.

Once you have a cone, you will know the **angle** of the circle it was made from. And you can measure make different quantities such as the cone's **height**.

How will **height** be related to **angle**?

- Predict: What do you think the relationship will look like?
- Record measurements of **height** and **angle**, using cones cut from circle sectors with the same radius.
- Plot **height** against **angle**.
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.

Making a Cone

Cones are delicious geometrical solids. Working with cones forces you to integrate your visualization skills with algebra and common sense.

Cones are easier to deal with than spheres, both algebraically and mechanically. For example, you can calculate the volume of a frustum of a cone by subtraction; and you can make a cone from a flat piece of paper.

Materials

You will need rulers, protractors, copies of the template (page 55ppp), scissors, tape, and long thin things such as broom straws or skewers.

First of all, you don't have to use the template to make the cones. But it may be a convenient shortcut to making large circular arcs.

It's challenging to measure the **height** of a cone accurately. We have found that when we roll up paper to make a cone, there is usually a small hole at the tip. You can insert a broomstraw or a thin skewer into that hole and use it like a dipstick to find the height, measuring the "dipped" part of the straw with a ruler.

Volume Extension

As an extension, you could look at the volume of a cone. To measure that, you could use rice or lentils and graduated cylinders—but be prepared for food on the floor. We have not been brave enough to make the cones of waxed paper and use water.

If you do volume, note that you can give students an extra challenge: find the **angle** for which **volume** is a maximum.

Different Radii

If you make your own cones, you can study height or volume as a function of radius. It would then be useful to control for angle; an easy way is to make only "equilateral" cones, that is, cones made from semicircles.

The Answer

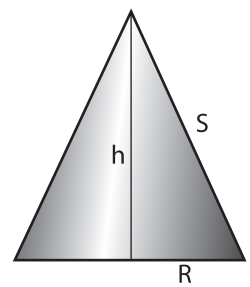
The key insight students need to have is that the *circumference* of the base of the final cone is equal to the *arc length* of the original sector.

If you measure the angle is θ in degrees, and the radius of the big circle is S, that arc length C is

$$C = 2\pi S \frac{\theta}{360}.$$

The radius R of the base of the cone is then

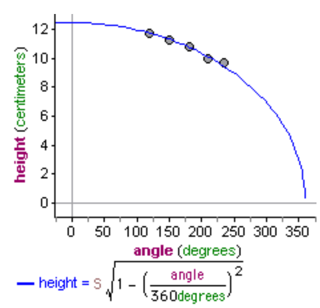
$$R = \frac{C}{2\pi} = \frac{S\theta}{360}.$$



Now if we look at a cross-section of the cone, we see a right triangle with R as the base and S—the slanty side of the cone, *which is the original radius*—as the hypotenuse. The height is just the other leg:

$$h = \sqrt{S^2 - R^2} = \sqrt{S^2 - S^2(\theta/360)^2} = S\sqrt{1 - (\theta/360)^2}.$$

Here is a sample graph. It would be good to get a wider range of angles:



24. Arctangent

In this activity, you will construct and measure a bunch of right triangles with a common base. Then you will compare the angle to the length of the changing side. You need paper, a ruler, and a protractor.

What to Do

Draw a number of triangles that share a 5-centimeter base. (See the illustration.) Measure the **angle** at the base of the triangle and the length of the **opposite** side.

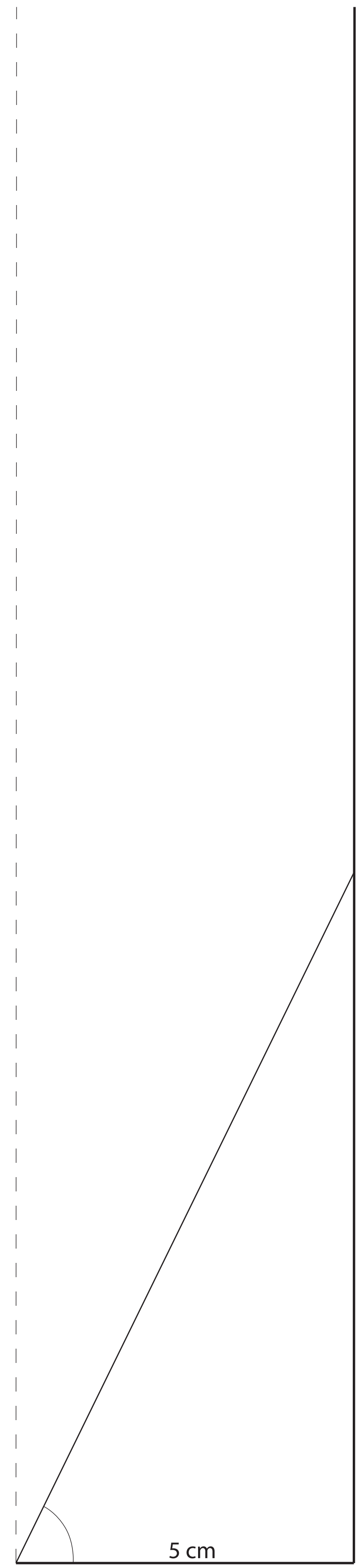
How will **angle** be related to **opposite**?

- Predict: What do you think the relationship will look like?
- Record your measurements of **opposite** and **angle**. Use a protractor to measure **angle** in degrees.

  The **angle** is the one between the base and the hypotenuse

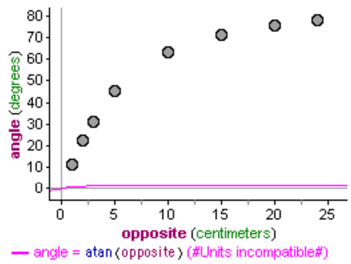
  Measure the length of the **opposite** side.

  Be sure you make your triangles so that **angle** ranges as widely as possible.
- Plot **angle** against **opposite**. (So **opposite** goes on the *horizontal* axis.)
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter.
- Be sure you can explain why the form of the function you used makes sense.



Arctangent

Here is what your data probably look like:



You can tell from the title of the activity that this is about arctangents. So if you put **opposite** on the horizontal axis, you can use **Plot Function** to plot **angle = atan(opposite)**. See the function waaaay down at the bottom? You need to multiply it by a constant, so

- stick a slider value **K** in the function so it's **angle = K*atan(opposite)**. But how big should **K** be?

Think about limiting cases. When **opposite** is really really huge, **angle** gets closer and closer to 90 degrees. Let's see that asymptote.

- Use **Plot Function** to plot **angle = 90**, which is a horizontal line.

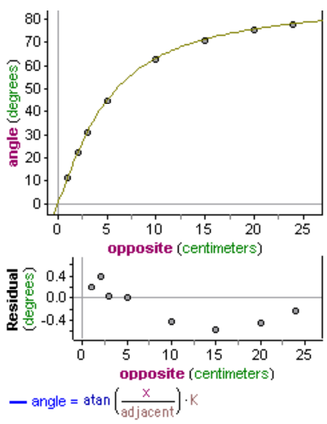
- Adjust **K** so that your model gets close to the new line but never crosses it. You may need to change the horizontal scale of the plot to get higher values, and the vertical scale to zoom in.

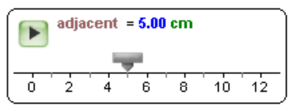
The shape is not right, so you need at least one more parameter. You just did a vertical stretch; you need a horizontal stretch as well. Try making a slider **A** and using **angle = K*atan(opposite/A)**. Residuals will help you adjust **A** and **K**. Then figure out what they mean.

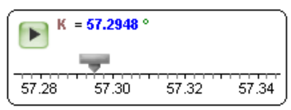
If you switch the axes, you get the tangent function. It's an interesting question why it is or is not OK to do that.

Note that can do secant with the same setup by measuring the hypotenuse.

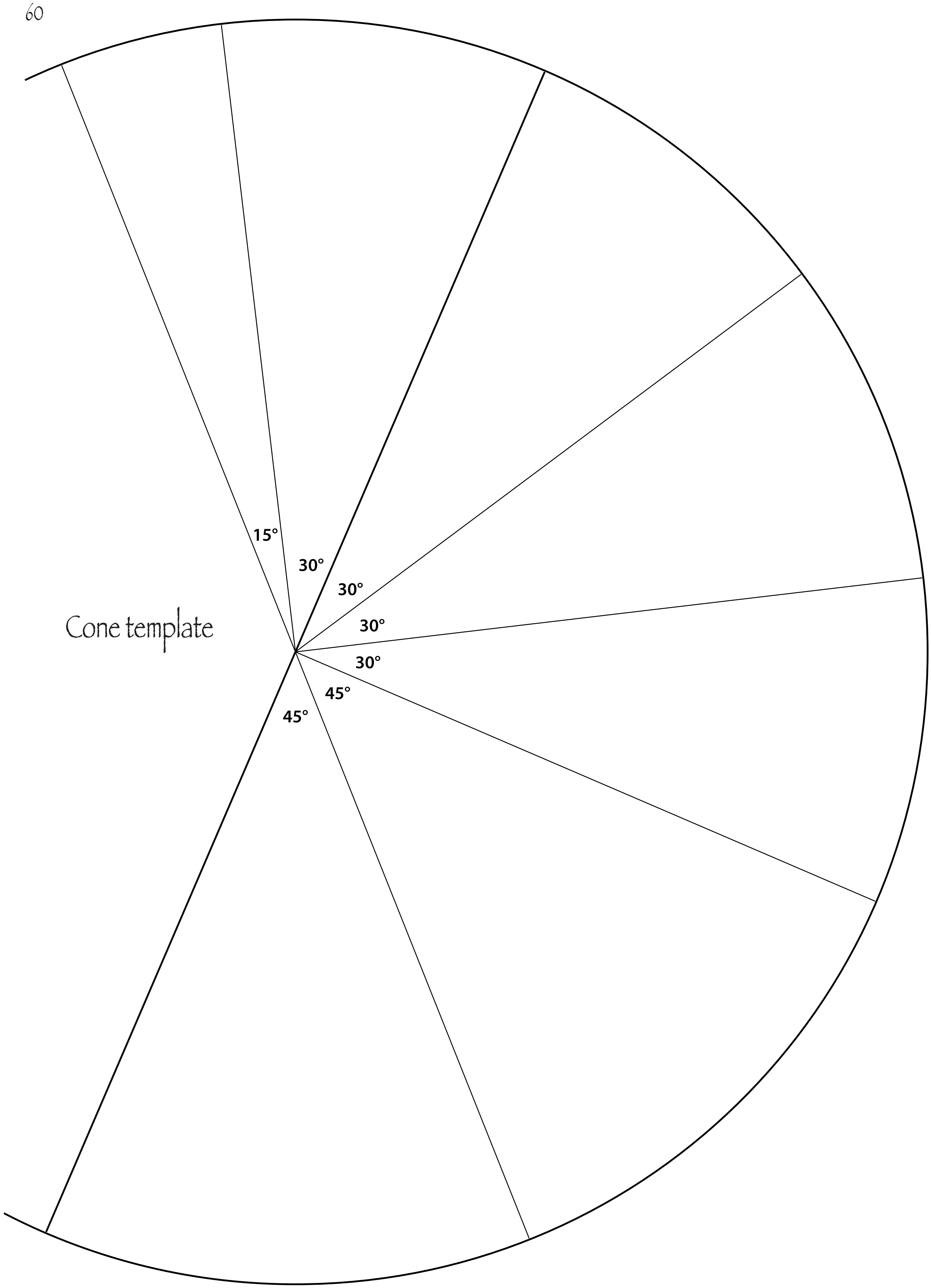
If you use two parameters, the vertical and horizontal stretch, it's interesting to see how the residuals move differently. Contrast this with using two parameters for a parabola or a line, where you discover that they are redundant. Here they aren't; the way the shape changes is different stretching horizontally than vertically.







Cone template



25. Sines of the Times

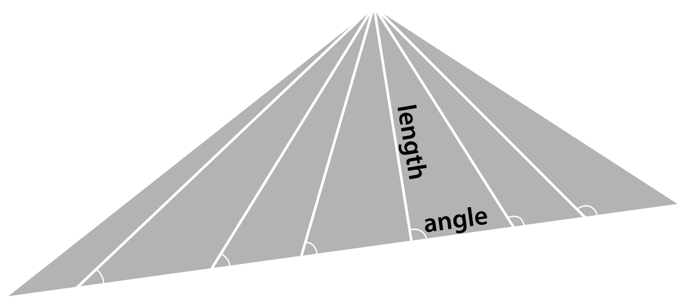
In this activity, you will compare the lengths of segments to related angles in a family of triangles. As the title suggests, you can use the law of sines to help you analyze this situation.

What to Do

First, you need to prepare your triangle:

- Cut or fold a large triangle out of a sheet of paper. Its exact shape does not matter.
- Pick the largest angle and draw segments radiating out from it, extending to the opposite side. Draw at least six, spread over the whole length of that opposite side.
- Mark an angle where each segment strikes the side. Mark the angle on the same side of the segment, as in the illustration. That is, all of the angles "face right."

Now you are ready to measure **angle** (the size of each angle) and the **length** of the corresponding segment.



How will **length** be related to **angle**?

- Predict: What do you think the relationship will look like?
- Record measurements of **length** and **angle**.
- Plot **length** against **angle**.
- Find and explain a mathematical function that fits the points. Be sure you can explain the meaning of any parameter. The Law of Sines could help you, but is not essential.
- Be sure you can explain why the form of the function you used makes sense.

Students will need rulers and protractors.

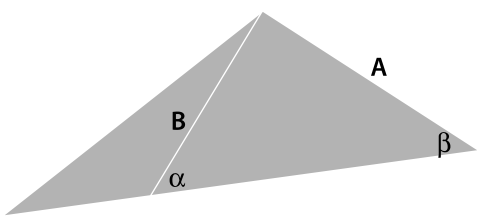
The Answer

There are several good approaches. One elegant strategy is to drop a perpendicular from the busy vertex of your diagram to the opposite side. Call that length *L*. If *x* is the length of one of the segments, then $L = x\sin\theta$. Solve for *x* to get:

$$x = \frac{L}{\sin\theta}$$, or, in our notation, $$\textbf{length} = \frac{\mathbf{K}}{\sin(\textbf{angle})},$$

where **K** is a free parameter. From our derivation, we also know its meaning: it's the length of the perpendicular, the shortest distance from the vertex to the opposite side.

You could also use the Law of Sines. In the diagram, notice that the angle β and the side **A** are the same for all of the triangles.



Our **length** is side **B**, our **angle** is angle α. Using the law of sines,

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$, we get $$B = \frac{A\sin\beta}{\sin\alpha}.$$

This is equivalent to the other result, since $A\sin\beta$ is also the length of that vertical.

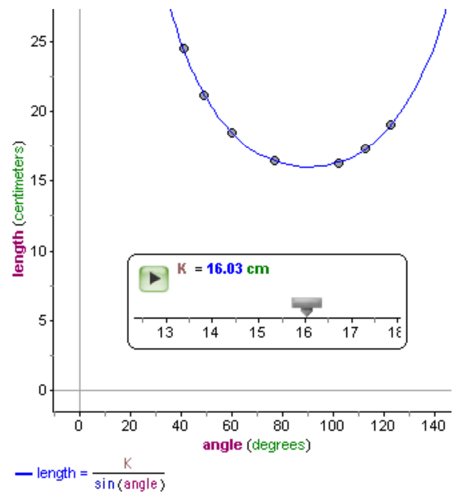
Which Axis?

In many of these activities, you could put either variable on the vertical axis depending on how you constructed whatever it is you are measuring. But here, the length of the segment depends on the angle, not the other way around. If you know the angle, you can figure out the length, but if you know the length, there might be two possible angles. The inverse is not a function.

This can be confusing because we pick a point on the base of the triangle, then draw the segment, then measure both the length and the angle. So really *both* depend on the position of the point.

Results

Here is an example of a good result:



Comments

If you don't analyze the situation, and just look at the graph, it's reasonable to wonder if it's quadratic. In fact, a quadratic function does a pretty good job of modeling the points near the vertex.

It's instructive to look at residuals from both a quadratic and from this "cosecant" relationship. The quadratic gets worse and worse the farther you get from the vertex. In fact, as you look at extreme cases—such as an angle greater than 180°—you can tell that a quadratic doesn't make sense.

You could linearize the situation by making a new column and calculating **(1/sin(angle))**. Plotting that against **length** should give a straight line.